Generalized model of second order parametric filter

Abstract In this paper the method of determining response to any excitation of some class second order filters with non-periodically variable parameters has been proposed. The variability of the parameters has been described by series of exponential functions. The restrictions of parametric functions which are guarantying correctness of this method have been presents. The results have been illustrated by the examples.

Streszenie W artykule została zaproponowana metoda wyznaczania odpowiedzi na dowolne wymuszenie pewnej klasy filtrów o zmiennych w czasie parametrach drugiego rzędu z parametrami zmiennymi nieokreśle w. Funkcje parametryzujące zostały opisane szeregami funkcji ekspONENTALNYCH. Przyjęte warunki przebiegu zmienności funkcji parametryzujących gwarantują poprawność stosowanej metody. Wyniki zostały zilustrowane przykładami. (Uogólniony model filtru parametrycznego drugiego rzędu).

Keywords: second order parametric system, LTV filter

Introduction

Linear systems with time-varying parameters (LTV) are described by linear differential equations with variable coefficients [1], [2], [4]. The certain method of determination of these equations having good physical interpretation consists in variation of coefficients of differential equations corresponding to transfer functions of LLS (linear lumped stationary) systems [2]. In the case of a low-pass second order LTV filter the equation can be presented in the following form [3], [7]:

\[ y''(t) + 2\sigma(t)y'(t) + \omega^2(t)y(t) = x(t) \]

where: \( y(t) \) – filter response to excitation \( x(t) \), \( \sigma(t) \) – time-varying angular frequency, \( \omega(t) \) – time-varying attenuation ratio.

In accordance with differential equations theory [3], [8] analytical solutions to equation (1) exist when the fundamental solutions to homogenous equation corresponding to equation (1) are known. Determination of fundamental solutions to such equations is possible only for special cases of parameters variability and requires knowledge about special functions of mathematical physics [3]. The mentioned above problem has been considered in earlier works [5], [7]. In the case of arbitrary non periodical functions \( \sigma(t) \), \( \omega(t) \) (compare with eq. (1)) the determination of fundamental solutions can be carried out using the method of successive approximations. The results obtained in this way are complex and difficult in interpretation.

In this article an analytical-numerical method of determination the fundamental solutions to equation (1) has been proposed [6]. The linear combination of obtained solutions can be interpreted as the LTV filter response to zero initial conditions with excitation \( f(t) \equiv 0 \). In the next step fundamental solutions the solution to non-homogenous system (1) have been determined. The entire solution to equation (1) will represent the complete mathematical model of LTV filter described by parametric differential equation (1).

Formalization of analytical-numerical method

It has been assumed that functions \( \sigma(t) \) and \( \omega(t) \) representing the varying coefficients of equation (1) and further referred to simply as the parametric functions are expressed by formulae:

\[ \sigma(t) = \sum_{k=0}^{n_1} \sigma_k \exp(-\beta_k t), \beta_k > 0, \sigma_k, \beta_k \in \mathbb{R}, \]

\[ \omega(t) = \sum_{k=0}^{n_2} \omega_k \exp(-\alpha_k t), \alpha_k > 0, \omega_k, \alpha_k \in \mathbb{R}^+ \]

The functions defined above are expressed by the sum of functions with limited energy and constant additive components and it is representation of non-periodical functions with are reached the steady values \( \omega_k \) in the time \( t \to \infty \). It can be proved [1], [5] that parametric filters described by (1) with earlier assumed conditions are asymptotic stable. After sufficiently long time the properties of LTV filters are identical as properties of low pass LLS filters with cut-off angular frequency \( \omega_0 \) and attenuation ratio \( \sigma_0 \). One can expressed equation (1) in following form:

\[ y''(t) + f(t)y'(t) + \left( f(t)g(t) - \omega^2(t) + g'(t) \right)y(t) = 0 \]

thus:

\[ f(t) = 2\sigma(t)\omega(t) \]

\[ g(t) = 2\sigma(t)\omega(t) + \omega^2(t) \]

where \( g(t) \) is some unknown function. It can be proved [3], that particular solutions to equation (6) are expressed by formulae:

\[ y_1(t) = \exp\left( \int g(t)dt \right) \]

\[ y_2(t) = \exp\left( -\int g(t)dt \right) + \int \exp\left( -2\int [\sigma(t)\omega(t) - g(t)]dt \right) \]

So the problem of solving equation (6) has been reduced to solving the Riccati equation in the following form:

\[ g'(t) = g^2(t) - 2\sigma(t)\omega(t)g(t) + \omega^2(t) \]

with function \( g(t) \) as a variable. For earlier assumptions (compare equations (2) and (3)) the analytical solutions to Riccati equation are unknown [3], [8] and can be found only by using numerical methods.

Algorithm description

The block diagram of proposed method of analysis of second order LTV filter has been shown in figure 1. The algorithm has been implemented in MATLAB™

First, Riccati equation (9) is solved for given input data using a numerical method. The solution is approximated by the exponential function series:

\[ g(t) = \sum_{k=0}^{n_2} G_k \exp(-\gamma_k t), G_k > 0, \gamma_k > 0, \gamma_0 = 0 \]

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The integral occurring in equation (12) should be computed using numerical methods or by series expansion of $\exp(t)$ function.

In the second phase one can obtain the solution to second order differential equation (1) which can be expressed as the system of two first order equations:

$$y_1(t) = \exp(G_0t - \sum_{k=1}^{n_y} G_k \exp(-\gamma_k t)), \quad y_2(t) = \exp(G_0t - \sum_{k=1}^{n_y} G_k \exp(-\gamma_k t)).$$

Taking into consideration the Wronski matrix (matrix of fundamental solutions and its derivatives) [1]:

$$W(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix},$$

one can determine the solution satisfies homogeneous system corresponding to the equation (1):

$$p(t) = p_c(t) + p_p(t),$$

where: $p_c(t)$ - vector of particular solutions:

$$p_p(t) = \int_0^t W(t) W^{-1} (\tau) x(\tau) d\tau.$$
The solution to Riccati equation (4) for assumed coefficients in case I and its approximation have been presented in figure 4.

Table 3. Coefficients of functions $g(t)$

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<tr>
<th>przykład</th>
<th>$g_0$</th>
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Numerical solution to equation (4) has been approximated by series of exponential functions (10). The coefficients of approximating formula are given also in table 3. Based on knowledge of function $g(t)$ one can find fundamental solution to equation (4) and one can determine the step response. The results of computing basing on equation (19), for zero initial conditions have been presented in figure 5.

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For comparison of result the response of classical LLS filter with constant parameters $\alpha_0=1$, $\omega_0=2$ rad/s have been drafted also.

Conclusions

The proposed model of generalized parametric filter with non-periodically variable parameters allows to determine the system response to any excitation. Nevertheless, this method requires the numerical aided solving of an auxiliary Riccati equation.

This model will be used in the future for optimization of dynamic properties of LTV systems and improving their characteristics comparing with LLS sections. The first and the second order sections are elementary blocks which enable building more complex filters with different properties. It allows to find optimal structures and determine values of parameters for minimum settling time.

In the stationary state the sections with variable parameters are equivalent to LLS sections and the advantages of proposed approach consist in improvement of dynamic properties.

LITERATURA


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