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Stationary action principle and particular solutions for long line equations

Abstract. *The recently developed by both authors a method for deriving telegraph equation from the stationarity principle of the action functional $I[u]$, under a particular form of the non-commutativity of operations od partial differentiation and taking variation, is further elaborated. The variational principle applied for field equations gives the availability of analytical or approximate solution of the equations. In the paper particular forms of solutions of the long line equation are searched.*

Streszczenie. *W artykule kontynuuje sie ostatnio rozwijana przez dwójke autorów metode wyprowadzania równań linii długiej z zasady stacjonarności działania $I[u]$ przy szczególnej postaci nieprzemienności operacji różniczkowania i brania wariacji. Zasada wariacyjna zastosowana do równań pola pozwala uzyskać warunki na analityczne albo przybliżone rozwiązania. W artykule poszukuje sie rozwiązań szczególnych i przybliżonych.*
*(**Zasada stacjonarnosci działania i rozwiązania szczególne równań linii długiej**)*

Keywords: telegraph equation, stationary action principle, long line equation, Lagrangian density, variational principle
Słowa kluczowe: równanie telegrafistów, zasada stacjonarności działania, równanie linii długiej, gęstość Lagrange'ianu, zasada wariacyjna

Introduction

The existence of a variation principle for given field equations has several advantages. The main refers to availability of analytical or approximate solution of the equations. In many cases the variational solution constitutes good approximation of the true one. This justifies the use of variational methods in treating complicated problems, like those involving irreversible phenomena, which cannot be solved directly.

Historically, at the beginning the classical Lagrange and Hamilton's formalisms were formulated for the point mechanics problems. Accordingly, if a dynamical system is described by the vector-valued coordinate $\boldsymbol{q}$ and the Lagrangian $L = T - V$, where T and V are, respectively, the kinetic and potential energy, then one formulates the variational principle of the dynamical system by requiring that between all curves $\boldsymbol{q} = \boldsymbol{q}(t)$ in a configuration space $\mathcal{V}$ the actual path (i.e. the solution of the system) is that which makes the action integral

$$I = \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\, dt \tag{1}$$

stationary. Taking the first variation $\delta \boldsymbol{q}$ subject to the conditions $\delta\boldsymbol{q}(t_1) = \delta\boldsymbol{q}(t_2) = 0$ the stationarity of the action requires $\delta I = 0$, which is equivalent to the Euler-Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0\,, \tag{2}$$

provided the classical commutative rule

$$\delta\dot{\boldsymbol{q}} = \frac{d}{dt}\delta\boldsymbol{q} \tag{3}$$

holds.

Recently both authors invented in [1] a method for deriving long line equation (with vanishing resistance) from the principle of stationarity of a particular action functional I, known for the conservative system, namely the wave equation. The main idea of the derivation is based on the observation, that for non-conservative systems and irreversible processes the variations of partial time and/or spatial derivatives of a field is different from the partial time and/or spatial derivative of the variation of the field, respectively [2]. This observation is based on the idea of Vujanović from the earliest 70'st [3, 4]. Hence from the action integral containing the density of a Lagrangian and known for a conservative system, equations of a non-conservative system is obtained by the variational principle, provided a particular form of the non-commutativity of operations is assumed.

The paper is formed as follow. In Sec. 2 the long line equation for general case is derived from the stationary action principle. Then a particular form of its solution is searched in Sec. 3. First the classical jump discontinuous solution for the line that meets the so-called Heaviside condition (15) is approximated by a smooth one. A numerical example is here presented. Then a particular exponential solution for voltage (18) is searched for an infinite long line. The derivation ends with a nonlinear ODE (28) which can be solved in parametric form. It will be done in the next paper. In Sec. 4 some remarks on the possible further consequences of the stationary action principle are given. The first deals with the balance of energy (29) derived from the principle by inventing the so-called global variation $\hat{\delta}u$. The second deals with the invariance of the action functional I under a group of transformations which relates this to a generalization of the Noether theorem to the non-conservative system.

Action principle for long line equation

The nonhomogeneous wave equation

$$F_L\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) := c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} - \kappa u = 0 \tag{4}$$

governs a conservative system. Hence its derivation in the form of Euler–Lagrange's (E–L's) equation from a Hamilton's principle is possible. In fact, assuming the density

$$L\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{\kappa}{2}u^2 \tag{5}$$

we get for the vanishing first variation

$$\delta I = \delta\int_{t_1}^{t_2}\int_{x_1}^{x_2} L dx dt = \int_{t_0}^{t_1}\int_{x_1}^{x_2}\{\{L\}_{E-L}\}\delta u dx dt\,, \tag{6}$$

where

$$\{L\}_{E-L} := \frac{\partial L}{\partial u} - \frac{\partial}{\partial x_j}\frac{\partial L}{\partial(\frac{\partial u}{\partial x_j})} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\frac{\partial u}{\partial t})}\,,$$

the E-L's equation in the form of Eq. (4), provided $\delta u(t_1) = \delta u(t_2) = 0$ and

$$\left[\delta, \frac{\partial}{\partial t}\right] u := \delta \frac{\partial}{\partial t} u - \frac{\partial}{\partial t} \delta u = 0 \tag{7}$$

hold.

Let us pass to long line equations which form a non-conservative system. The two equations

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} &= -R_0 i(x,t) - L_0 \frac{\partial i(x,t)}{\partial t} \\ \frac{\partial i(x,t)}{\partial x} &= -G_0 u(x,t) - C_0 \frac{\partial u(x,t)}{\partial t}, \end{aligned} \tag{8}$$

with suitable voltage and current functions $u(x,t)$ and $i(x,t)$ govern the problem. After differentiation and eliminating $i(x,t)$ we get

$$\frac{\partial^2 u(x,t)}{\partial x^2} = L_0 C_0 \frac{\partial^2 u(x,t)}{\partial t^2} + (R_0 C_0 + G_0 L_0) \frac{\partial u(x,t)}{\partial t} + R_0 C_0 u(x,t), \tag{9}$$

where the resistance R_0 is different from zero. On the other hand after differentiation and eliminating $u(x,t)$, we get

$$\frac{\partial^2 i(x,t)}{\partial x^2} = L_0 C_0 \frac{\partial^2 i(x,t)}{\partial t^2} + (G_0 L_0 + C_0 R_0) \frac{\partial i(x,t)}{\partial t} + G_0 R_0 i. \tag{10}$$

In our previous paper [1] we put the conductance G_0 equal to zero, then the last equation has simplified to

$$\frac{\partial^2 i(x,t)}{\partial x^2} = C_0 L_0 \frac{\partial^2 i(x,t)}{\partial t^2} + C_0 R_0 \frac{\partial i(x,t)}{\partial t}. \tag{11}$$

Let us introduce the density of the Lagrangian in the form of (5) where c^2 and κ are substituted with

$$\alpha = (C_0 L_0)^{-1}, \kappa = \alpha R_0 C_0, \tag{12}$$

and let us admit the noncommutative rule (cf. (7))

$$\left[\delta, \frac{\partial}{\partial t}\right] u = -\gamma \delta u \tag{13}$$

with $\gamma = \alpha(R_0 C_0 + G_0 L_0)$.

Now if we define the action functional as previously and apply the Hamilton's principle $\delta I = 0$, then we get from (6)– under the boundary conditions $\delta u(t_1) = \delta u(t_2)$ for the first variation of u – the E–L's equation exactly as (10).

On the other hand, for the case of the current function, if we substitute the voltage $u(x,t)$ with the current $i(x,t)$ in the last density of Lagrangian L with new $\bar{\kappa} = \alpha G_0 R_0$ and use similar noncommutative rule (13), and apply the Hamilton's principle $\delta I = 0$, with (5), under the suitable boundary conditions for the first variation of i, the E-L's equation is the governing equation for the current $i(x,t)$ (11).

It is interesting to add that in the case of non-homogeneous physical (material) properties of the line, i.e. when $R(x), C(x), L(x)$ and $G(x)$ are functions of x, the E–L's equations can be derived from the Hamilton principle by putting appropriate γ in (13).

Particular solutions

It is well-known that one of possible solutions of the long line equation is (cf. [5] p.271)

$$\begin{aligned} u(x,t) &= e \exp(-\alpha_1 x) H(t - \tau_x), \\ i(x,t) &= \frac{e}{Z_c} \exp(-\alpha_1 x) H(t - \tau_x), \end{aligned} \tag{14}$$

with $H(t - \tau_x)$ as the Heaviside (unit jump) function, $\alpha_1 = (R_0 G_0)^{1/2}$ and $\tau_x = x \alpha^{-1/2}$. The parameter α_1 is the so-called damping parameter. As previously(cf. Eq.12) the parameter $\alpha = (C_0 L_0)^{-1}$. Here Z_c is the so-called wave impedance and e is a constant voltage supply. This solution is valid for the line that meets the so-called Heaviside condition which requires the constant proportion:

$$\frac{R_0}{L_0} = \frac{G_0}{C_0}. \tag{15}$$

Let us try to approximate this solution by introducing the so-called binary sigmoidal (logistic) function

$$\sigma(z) = \frac{2}{1 + \exp -\beta z} - 1, \text{ if } z \geq 0 \text{ and } \sigma(z) = 0, \text{otherwise}, \tag{16}$$

with some $\beta > 0$. Notice that the parameter β in the function $\sigma(z)$ governs the slop of the curve which approximate the Heaviside function: for small, i.e. less than one, β the slop is small, however the large β diminishes substantially the intermediate region in which the values of σ differs from those two limit values: 0 and 1. With σ let us try to approximate the solutions (14) assuming

$$\begin{aligned} u(x,t) &= e \exp(-\alpha_1 x) \sigma(t - \tau_x), \\ i(x,t) &= \frac{e}{Z_c} \exp(-\alpha_1 x) \sigma(t - \tau_x). \end{aligned} \tag{17}$$

If no other conditions are formulated then only consistence relation is $Z_c = C_0 \alpha^{-1/2} = (\frac{L_0}{C_0})^{1/2}$. The presentation of the numerical solutions for particular set of material parameters and two values of $\beta = 0.5$ and $\beta = 2$ are given on the figures below. The time period was 20 second and the line length 200 km in both cases.

Now let us use the stationary action principle to find another particular solution. Let us suppose that to one face of the infinite long line at $x = 0$ a voltage jump is applied and introduce the generalized coordinate $q(t)$ for the voltage function

$$u(x,t) = e \exp(-x/q(t)). \tag{18}$$

Now we are going to use our stationary action principle for determining the form of $q(t)$. As in Sec. 3 the Lagrangian density (5) will be used with $c^2 = \alpha$ and the first variation (6) of the action functional I with the form (18) substituted. Integrating with respect to x one has (the similar action was made in [6])

$$\bar{I}[q] = \int_{t_1}^{t_2} \left[(\frac{1}{8})(\frac{\dot{q}^2}{q}) - \frac{\kappa q}{2} - \frac{\alpha}{4q} \right] dt. \tag{19}$$

Assuming the first variation of $\bar{I}[q]$ vanishes with the non-commutative law (13), written now for q, namely $\left[\delta, \frac{\partial}{\partial t}\right] q = -\gamma \delta q$, we end us with the E–L's equation of the form

$$\frac{1}{4} \frac{d}{dt} (\frac{\dot{q}}{q}) + \frac{1}{8} (\frac{\dot{q}}{q})^2 + \frac{\gamma}{4} \frac{\dot{q}}{q} - \frac{\alpha}{4q^2} + \frac{\kappa}{4} = 0. \tag{20}$$

It is evident that one possible solution is $\dot{q} = 0$, and consequently $q^2 = \alpha\kappa$. From this we get

$$u(x,t) = E \exp(-x (R_0 G_0)^{-1/2}). \tag{21}$$

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R0 = 77*10^{-3};
G0 = 4 * 10^{-8};
L0 = 24 * 10^{-3};
C0 = 12.5* 10^{-9};
α = 1 / (L0 * C0);
α1 = Sqrt[R0 * G0];
e = 400 * 10^3;
β = 2;
U[x_, t_] = e * Exp[(-α1 * x)] * (2 / (1 + Exp[-β * (t - x * α^(-1/2))]) - 1);
Plot3D[U[x, t], {x, 0, 200}, {t, 0, 20}]
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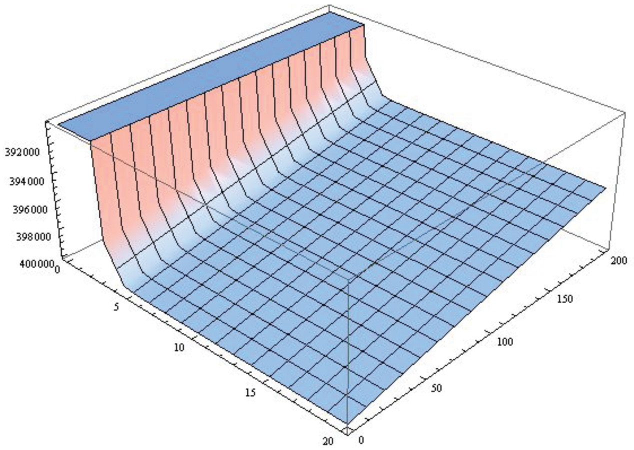

Fig. 2. The graph of the voltage with $e = 400$ kV and $\beta = 2$ according to (17).

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e = 400 * 10^3;
β = 0.5;
U[x_, t_] = e * Exp[(-α1 * x)] * (2 / (1 + Exp[-β * (t - x * α^(-1/2))]) - 1);
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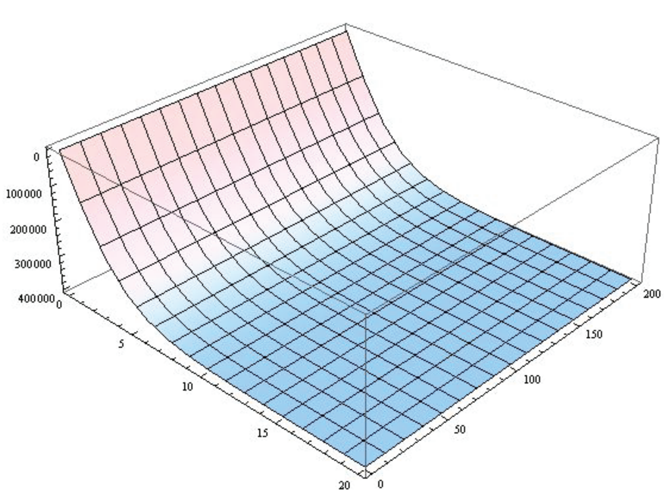

Fig. 1. The graph of the voltage with $e = 400$ kV and $\beta = 0.5$ according to (17).

We are rather interested in a non-constant solution. If $\dot{q} \neq 0$ the function $q = q(t)$ is invertible and we have t as a function of q. Then we can write that a function $v(q)$ exists such that $\dot{q} = v(q)$. Calculate the second time derivative of q, we get

$$\ddot{q}(t) = v'(q)v(q) \,. \tag{22}$$

Consequently, Eq.(20) can be transformed to

$$v'(q)v(q) - \frac{1}{2q}v(q)^2 + \gamma v(q) + \kappa q - \frac{\alpha}{q} = 0 \,. \tag{23}$$

It is an Abel's equation of the second kind (cf.[7], p.411). By defining $\mathcal{E}(q) = \exp(-\int \frac{1}{2q}dq) = (q)^{-1/2}$ and introducing new function

$$y(q) = v(q)\mathcal{E}(q) \,, \tag{24}$$

the last Eq.(23) can be transformed into

$$yy' = -\gamma\mathcal{E}y - (\kappa q - \frac{\alpha}{q})\mathcal{E}^2 \,. \tag{25}$$

Let us put

$$z(q) = y(q) + F(q) \,, \tag{26}$$

with $F(q) = -\int(-\gamma(q)^{-1/2})dq = 2\gamma q^{1/2}$, then in the new dependent variable $z(q)$ we get

$$(z(q) + F(q))z'(q) = \frac{\alpha}{q^2} - \kappa \,. \tag{27}$$

Now if we invent the new variable $\zeta = -\int(\kappa q - \alpha/q)\mathcal{E}^2 dq = -\alpha/q - \kappa q$ and express z in terms of ζ, i.e. $\eta(\zeta) := z(q(\zeta))$, then from Eq. (27) we receive

$$(\eta(\zeta) + F(q(\zeta)))\eta'(\zeta) = 1 \,. \tag{28}$$

The last equation can be solved in a parametric form ([7]). It will be the subject of the next paper.

Conclusions

The paper is a continuation of our previous paper [1] in which the further development of the stationary action principle for the long line equation is performed. The case of of non-vanishing resistance and conductance is considered, and the Lagrangian density function is here proposed. Particular solutions are searched in stationary and non-stationary cases. In the non-stationary case the first solution is a smooth approximation of the solution of impedance matching. The searching the second solution the principle of stationary action is used for determining the unknown generalized coordinate $q(t)$ appearing in the exponential form of the voltage function.

It is interesting to add that by introducing the so-called global variation $\hat{\delta}u$ as the sum of the present variation δu and the time variation δt, i.e.

$$\hat{\delta}u = \delta u + \delta t \,,$$

we may obtain from the same stationary action principle the second Euler-Lagrange equation in the form of the energy conservation (balance) law for the long line, namely

$$\frac{1}{2}\int_a^b \left(\dot{u}^2 + \alpha(\nabla u)^2 + \kappa u^2\right)\Big|_{t_1}^{t_2} dx = \int_{t_1}^{t_2} \left(\alpha \nabla u \dot{u}\Big|_a^b + \int_a^b \gamma \dot{u}^2 dx\right) dt \,. \tag{29}$$

The same balance law can be determined by generalizing the Noether theorem responsible for the symmetric properties of action functional of the system. This can be regarded as a particular generalization of the Noether theorem to the non-conservative system, confined to the voltage of the long line. The full derivation of our generalization of the

Noether theorem for general non-conservative (dissipative) system is done in [9].

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REFERENCES

[1] B. Grochowicz, W. Kosiński. Lagrange's method for derivation of long line equations, *Acta Technica*, **56** (3), 331–341, 2011.
[2] W. Kosiński, P. Perzyna. On consequences of the principle of stationary action for dissipative bodies, *Arch. Mech.*, **64** (1), 95–106, 2012.
[3] B. Vujanović. On one variational principle for irreversible phenomena. *Acta Mechanica*, **19** (1974), 259–275.
[4] B. Vujanović. A variational principle for non-conservative dynamical systems. *ZAMM – Zeitschrift für Angewandte Mathematik und Mechanik*, **55**(6), (1975), 321–331.
[5] T. Cholewicki. *Elektrotechnika techniczna*, t.II, Wydawnictwa Naukowo–Techniczne, Warszawa, 1971.
[6] B. Vujanovíc. An approach to linear and nonlinear heattarnsfer problem using a Lagrangian. *AIAA Journal*, **9** (10), (1971), 131–134.
[7] E. Kamke. Differential Gleichungen. Lösungsmethoden und Lösungen. Bd.I. Gewönliche Differentialgleichungen. B.G. Teubner, Leipzig, 1959.
[8] E. Noether. Invariante Variationsprobleme. *Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys.*, Klasse 1918 (1918), 235–257.
[9] W. Kosiński and P. Perzyna. Symmetry properties of action functional for non-conservative systems. *Under preparation.*

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