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Approximate BEM analysis of time-harmonic magnetic field nearby current-lines in thin shield

Streszczenie. *Rozpatruje się możliwość zastosowania przybliżonego modelu obliczeniowego do analizy pola magnetycznego przewodów z prądem umieszczonych wewnątrz cienkościennej przewodzącej osłony. W tym celu wykorzystuje się metodę elementów brzegowych (MEB), lecz ze względu na założenie o małej grubości osłony pole wewnątrz niej wyraża się teoretyczną zależnością przybliżoną. Pozwala to zmniejszyć nakład obliczeń, a także uniknąć ewaluacji całek prawie osobliwych.*

Abstract*. This paper focuses on a possibility of using an approximate computational model in the analysis of the magnetic field around a current lines enclosed in a thin conductive shield. The model uses the boundary element method (BEM), but the field in the shield is expressed with an approximate expression due to small thickness of the shield. This reduces the computational effort and avoids evaluating the nearly-singular integrals. (****Przybliżona analiza harmonicznego pola magnetycznego przewodów z prądem w cienkościennej osłonie za pomocą MEB****).*

Słowa kluczowe: MEB, harmoniczne pole magnetyczne, linie osłonięte, struktury cienkościenne.
Keywords: BEM, time-harmonic magnetic field, enclosed lines, thin bodies.

Introduction

Analysis of magnetic field in current lines is a subject o many papers, i.e. [1-3] to name a few. Realistic configurations, like current lines enclosed in conductive shields, often require a use of numerical methods. If the shields are relatively thin, they can be troublesome in numerical analysis. In FEM, for example, they require a very fine mesh, in BEM – result in nearly singular integrals, the numerical evaluation of which can be very inaccurate. Therefore, thin shields require special treatment [4-9]. This paper show one of possible approaches.

Problem description and governing equations

A set of long, parallel wires with time-harmonic currents $\underline{I}_1$, $\underline{I}_2$, …, $\underline{I}_K$ of angular frequency ω is enclosed in a thin conductive shield Ω_s, the thickness of which, d, is small in relation to the cross-section dimensions (Fig. 1). The internal and external regions of the shield are Ω_i and Ω_e, respectively. All regions are assumed to be non-magnetic (μ_r = 1). The goal is to find out the magnetic field in the configuration.

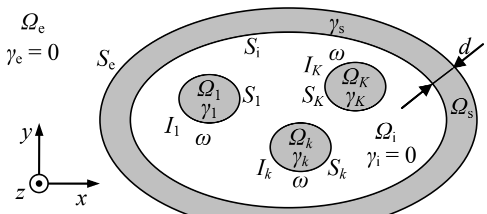

Fig. 1. Current lines enclosed in a thin shield

With z-axis along the wires, the magnetic field in the configuration can be expressed in terms of the z-component of the magnetic vector potential. In the particular expressions it fulfills the following equations

$$\text{(1)} \qquad \begin{cases} \nabla^2 \underline{A}^{(k)} - \kappa_k^2 \underline{A}^{(k)} = 0, & k = \mathrm{s}, 1, 2, ..., K, \\ \nabla^2 \underline{A}^{(k)} = 0, & k = \mathrm{i}, \mathrm{e}, \end{cases}$$

where $\kappa_k^2 = \mathrm{j}\omega\mu_0\gamma_k$. The continuity of the tangent components of magnetic field intensity results in the continuity of normal derivatives of $\underline{A}$ on each boundary. Since different potential gauge is used in each region (to make the Helmholtz equations homogeneous), the potential can be discontinuous on the boundaries so that

$$\text{(2)} \qquad \begin{cases} S_k : \ \underline{A}^{(\mathrm{i})} - \underline{A}^{(k)} = C_k, & k = 1, 2, ..., K, \\ S_\mathrm{i} : \ \underline{A}^{(\mathrm{i})} - \underline{A}^{(\mathrm{s})} = 0, \\ S_\mathrm{e} : \ \underline{A}^{(\mathrm{e})} - \underline{A}^{(\mathrm{s})} = C_\mathrm{s}, \end{cases}$$

where C_k and C_s are K + 1 constants corresponding to each conductive region. To determine them, K + 1 additional equations must be formulated. They are Ampère's laws for external contours Γ_s, Γ_1, Γ_2, …, Γ_K of the cross sections of each conductive region:

$$\text{(3)} \qquad \oint_{\Gamma_k} \frac{\partial \underline{A}^{(k)}}{\partial n} \mathrm{d}\Gamma = -\mu_0 \underline{\Theta}_k, \qquad k = \mathrm{s}, 1, 2, ..., K.$$

where $\underline{\Theta}_k$ is the total current through contour Γ_k (i.e. $\underline{\Theta}_s = \Sigma \underline{I}_k$, $\underline{\Theta}_1 = \underline{I}_1$, $\underline{\Theta}_2 = \underline{I}_2$, …, $\underline{\Theta}_K = \underline{I}_K$).

Conventional BEM model

The above mentioned equations can be solved by means of BEM [10-11]. Each of Eqs. (1), through its boundary integral equivalent, results in a set o algebraic equations, which can be written as matrix equation. They are as follows:

$$\text{(4)} \qquad \mathbf{H}_k^k \mathbf{A}_k^k = \mathbf{G}_k^k \mathbf{Q}_k^k, \qquad k = 1, 2, ..., K$$

for each of K wires,

$$\text{(5)} \qquad \mathbf{H}_\mathrm{i}^\mathrm{s} \mathbf{A}_\mathrm{i}^\mathrm{s} + \mathbf{H}_\mathrm{e}^\mathrm{s} \mathbf{A}_\mathrm{e}^\mathrm{s} = \mathbf{G}_\mathrm{i}^\mathrm{s} \mathbf{Q}_\mathrm{i}^\mathrm{s} + \mathbf{G}_\mathrm{e}^\mathrm{s} \mathbf{Q}_\mathrm{e}^\mathrm{s}$$

for the shield, and

$$\text{(6)} \qquad \begin{cases} \mathbf{H}_\mathrm{i}^\mathrm{i} \mathbf{A}_\mathrm{i}^\mathrm{i} + \sum\limits_{k=1}^{K} \mathbf{H}_k^\mathrm{i} \mathbf{A}_k^\mathrm{i} = \mathbf{G}_\mathrm{i}^\mathrm{i} \mathbf{Q}_\mathrm{i}^\mathrm{i} + \sum\limits_{k=1}^{K} \mathbf{G}_k^\mathrm{i} \mathbf{Q}_k^\mathrm{i}, \\ \mathbf{H}_\mathrm{e}^\mathrm{e} \mathbf{A}_\mathrm{e}^\mathrm{e} = \mathbf{G}_\mathrm{e}^\mathrm{e} \mathbf{Q}_\mathrm{e}^\mathrm{e} \end{cases}$$

for the internal and external non-conductive regions. In equations (4)÷(6), $\mathbf{A}_a^b$ and $\mathbf{Q}_a^b$ are column vectors of $\underline{A}$ and $\partial\underline{A}/\partial n$ on boundary S_a for region Ω_b, whereas $\mathbf{G}_a^b$ and $\mathbf{H}_a^b$ are the standard BEM matrices formed from the integrals of the fundamental solution for region Ω_b and boundary elements on boundary S_a.

The interface conditions on each of the boundaries take the following form:

$$\text{(7)}\qquad \begin{cases} \mathbf{A}_k^{\mathrm{i}} = \mathbf{A}_k^{k} + \mathbf{1}C_k, & \mathbf{A}_{\mathrm{e}}^{\mathrm{e}} = \mathbf{A}_{\mathrm{e}}^{\mathrm{s}} + \mathbf{1}C_{\mathrm{s}}, \quad \mathbf{A}_{\mathrm{i}}^{\mathrm{i}} = \mathbf{A}_{\mathrm{i}}^{\mathrm{s}}, \\ \mathbf{Q}_k^{\mathrm{i}} = -\mathbf{Q}_k^{k}, & \mathbf{Q}_{\mathrm{e}}^{\mathrm{e}} = -\mathbf{Q}_{\mathrm{e}}^{\mathrm{s}}, \qquad \mathbf{Q}_{\mathrm{i}}^{\mathrm{i}} = -\mathbf{Q}_{\mathrm{i}}^{\mathrm{s}}, \end{cases}$$

where k = 1, 2, …, K, and $\mathbf{1}$ is a column vector of ones of appropriate dimension. The negative signs in the last three relationships come from the assumed orientation of the normal vectors (outwards region Ω_b). Relationships (7) allow eliminating some of vectors from equations (6) so that

$$\text{(8)}\qquad \begin{cases} \mathbf{H}_{\mathrm{i}}^{\mathrm{i}}\mathbf{A}_{\mathrm{i}}^{\mathrm{s}} + \sum_{k=1}^{K}\mathbf{H}_k^{\mathrm{i}}\mathbf{A}_k^{k} + \sum_{k=1}^{K}\mathbf{1}_k C_k = -\mathbf{G}_{\mathrm{i}}^{\mathrm{i}}\mathbf{Q}_{\mathrm{i}}^{\mathrm{s}} - \sum_{k=1}^{K}\mathbf{G}_k^{\mathrm{i}}\mathbf{Q}_k^{k}, \\ \mathbf{H}_{\mathrm{e}}^{\mathrm{e}}\mathbf{A}_{\mathrm{e}}^{\mathrm{s}} + \mathbf{1}_{\mathrm{e}}C_{\mathrm{s}} = -\mathbf{G}_{\mathrm{e}}^{\mathrm{e}}\mathbf{Q}_{\mathrm{e}}^{\mathrm{s}}. \end{cases}$$

The discrete form of equations (3) are

$$\text{(9)}\qquad \begin{cases} \mathbf{L}_k\mathbf{Q}_k^{k} = -\mu_0 \underline{I}_k, \quad k = 1,2,...,K, \\ \mathbf{L}_{\mathrm{e}}\mathbf{Q}_{\mathrm{e}}^{\mathrm{s}} = -\mu_0 \sum_{k=1}^{K}\underline{I}_k, \end{cases}$$

where $\mathbf{L}_k$ is a row vector of lengths of elements belonging to boundary S_k.

Equations (4), (5), (8) and (9) can be formed into one system of equations, and solved with respect to $\mathbf{A}_a^b$, $\mathbf{Q}_a^b$ and C_k. The computational model, referred to as CBEM (conventional BEM) in the subsequent text, is correct for any shapes of wires and shield cross-sections. However, if the shield is thin, some integrals in matrices $\mathbf{G}_{\mathrm{i}}^{\mathrm{s}}$, $\mathbf{G}_{\mathrm{e}}^{\mathrm{s}}$, $\mathbf{H}_{\mathrm{i}}^{\mathrm{s}}$ and $\mathbf{H}_{\mathrm{e}}^{\mathrm{s}}$ are nearly singular, and therefore, require much computational effort to be evaluated with suitable accuracy.

BEM model for thin shield

Because the shield is assumed to be thin, there is hope that it can be treated in a special way. If the shield is thin enough, the shell between two corresponding boundary elements lying on S_{i} and S_{e} may be approximately regarded as a fragment of infinite plate. In such a plate, the general solution of the first of Eqs. (1) for the shell becomes

$$\text{(10)}\qquad \underline{A}^{(\mathrm{s})}(\zeta) \approx C_1 \cosh\kappa_{\mathrm{s}}\zeta + C_2 \sinh\kappa_{\mathrm{s}}\zeta\,,$$

where C_1 and C_2 are constants, and $0 \le \zeta \le d$. Assuming that $\underline{A}(0) = \underline{A}_{\mathrm{i}}$ and $\underline{A}(d) = \underline{A}_{\mathrm{e}}$, the constants can be evaluated so that

$$\text{(11)}\qquad \underline{A}^{(\mathrm{s})}(\zeta) = \frac{\underline{A}_{\mathrm{i}}\sinh\kappa_{\mathrm{s}}(d-\zeta) + \underline{A}_{\mathrm{e}}\sinh\kappa_{\mathrm{s}}\zeta}{\sinh\kappa_{\mathrm{s}}d}\,,$$

and therefore,

$$\text{(12)}\qquad \frac{\partial\underline{A}^{(\mathrm{s})}(\zeta)}{\partial n} = \kappa_{\mathrm{s}}\frac{-\underline{A}_{\mathrm{i}}\cosh\kappa_{\mathrm{s}}(d-\zeta) + \underline{A}_{\mathrm{e}}\cosh\kappa_{\mathrm{s}}\zeta}{\sinh\kappa_{\mathrm{s}}d}\,.$$

This leads to the following relationships:

$$\text{(13)}\qquad \begin{cases} \underline{Q}_{\mathrm{i}} = -\left.\dfrac{\partial\underline{A}^{(\mathrm{s})}}{\partial\zeta}\right|_{\zeta=0} = \sigma\underline{A}_{\mathrm{i}} + \tau\underline{A}_{\mathrm{e}}, \\ \underline{Q}_{\mathrm{e}} = \left.\dfrac{\partial\underline{A}^{(\mathrm{s})}}{\partial\zeta}\right|_{\zeta=d} = \tau\underline{A}_{\mathrm{i}} + \sigma\underline{A}_{\mathrm{e}}, \end{cases}$$

where

$$\text{(14)}\qquad \sigma = \kappa_{\mathrm{s}}\coth\kappa_{\mathrm{s}}d, \qquad \tau = -\kappa_{\mathrm{s}}\,\mathrm{csch}\,\kappa_{\mathrm{s}}d\,.$$

Equations (13) written in discrete form,

$$\text{(15)}\qquad \mathbf{Q}_{\mathrm{i}}^{\mathrm{s}} \approx \sigma\mathbf{A}_{\mathrm{i}}^{\mathrm{s}} + \tau\mathbf{A}_{\mathrm{e}}^{\mathrm{s}}, \qquad \mathbf{Q}_{\mathrm{e}}^{\mathrm{s}} \approx \sigma\mathbf{A}_{\mathrm{e}}^{\mathrm{s}} + \tau\mathbf{A}_{\mathrm{i}}^{\mathrm{s}}\,,$$

deliver approximate relationships between nodal values of $\underline{A}^{(\mathrm{s})}$ and $\partial\underline{A}^{(\mathrm{s})}/\partial n$ on boundaries S_{i} and S_{e} so that there is no need to use equation (5) anymore. The resulting model is referred to as ABEM (approximate BEM) in the subsequent paragraphs. When compared to CBEM, its system of equations does not contain $\mathbf{Q}_{\mathrm{i}}^{\mathrm{s}}$ and $\mathbf{Q}_{\mathrm{e}}^{\mathrm{s}}$, therefore, it has a smaller main matrix. There are no nearly singular integrals (for sufficiently regular boundary of the shield). Since the model uses approximate relationships, it should be uses with care. Numerical simulations show that it works the better the greater $|\kappa_{\mathrm{s}}|$ and the smaller d.

Numerical example

Both presented models, were implemented in Mathematica 7.0 and tested in various configurations and parameters. Boundary elements with constant field approximation and quadratic geometry approximation were used. Such elements are rather unusual, but they allow taking into account possible curvature of the boundary and retaining the simplicity of matrix generation.

As an example let us consider a shielded symmetrical three-phase line of arbitrary dimensions and cross-section shown in Figure 2. Each wire has circular cross-section of radius R_{w}, and the shield is a tube of internal and external radii R_{i} and R_{e}, respectively. Let us introduce the following auxiliary quantities: $d_{\mathrm{ww}} = \sqrt{3}R_1 - 2R_{\mathrm{w}}$ (distance between the closest points of wires), $d_{\mathrm{ws}} = R_i - (R_1 + R_{\mathrm{w}})$ (distance between the closest points of a wire and the shield), Δ_{w} − skin depth for the wire, Δ_{s} – skin depth for the shield, $\delta = (R_{\mathrm{e}} - R_{\mathrm{i}})/R_{\mathrm{i}}$ (the relative thickness of the shield).

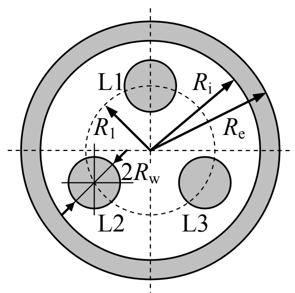

Fig. 2. Symmetrical shielded three-phase line

The results given in the subsequent paragraphs were obtained for the following discretization: each wire − 20 elements, both the internal and external boundary of the shield − 40 elements (the total number of boundary elements was 140). The computations were performed for different values of skin depth (Δ_{w} and Δ_{s}, which were assumed equal) and the relative shield thickness δ. Each computation was performed using three different computational models. The first one, referred to as CBEM1, is CBEM with the internal built-in Mathematica procedures to evaluate the nearly singular integrals. The same problem was solved with use of ABEM, but since the nearly singular integrals are absent then, the BEM integrals were evaluated with much quicker Gaussian quadrature of order 10. The same settings were used then in model CBEM, referred to as CBEM2.

The results obtained from CBEM1 were treated as the most exact of the three. In general, the CBEM1 was the most time consuming − it was nearly twice slower than CBEM2 and about 3÷4 slower than ABEM, which was the quickest. Figure 3 shows nodal values of $\nabla_n \underline{A}$ (its real part), which is the negative tangent component of magnetic flux density, for δ = 0.1, and Figure 4 – for δ = 0.01 for arbitrary values of parameters. Numerical tests indicate that if the shield is thick enough the results given by CBEM1 and CBEM2 are practically the same, and ABEM gives some small differences (Fig. 3). For smaller values of thickness the differences between the three models becomes smaller, and practically indistinguishable. For very small thickness ABEM produces practically the same results as CBEM1, whereas CBEM2 leads to significant errors (Fig. 4).

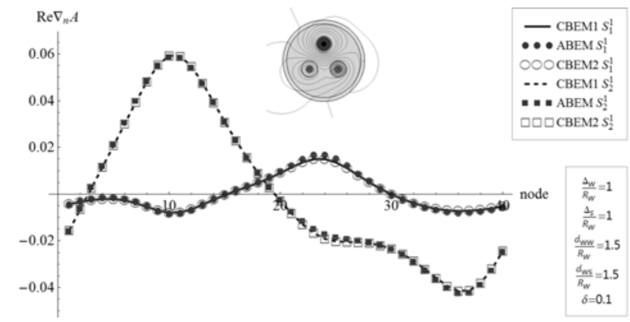

Fig. 3. Nodal values of $\mathrm{Re}\nabla_n \underline{A}$ on external (S_1^1) and internal (S_2^1) boundary of the shell for δ = 0.1 and Δ_s/R_w = 1

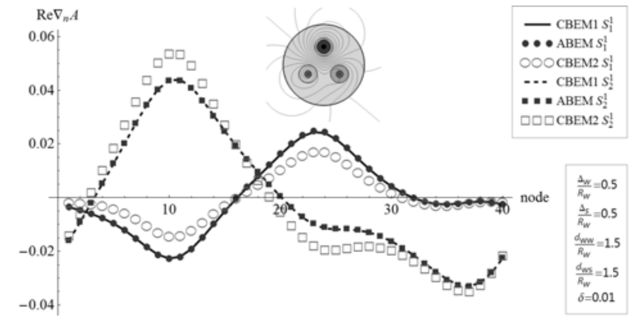

Fig. 4. Nodal values of $\mathrm{Re}\nabla_n \underline{A}$ on external (S_1^1) and internal (S_2^1) boundary of the shell for δ = 0.01 and Δ_s/R_w = 0.5

Figures 5 and 6 shows active and reactive power penetrating the shield (δ = 0.1 and 0.01, respectively) per unit length and divided by $I^2 R_{DC}$, where I is the line current and R_{DC} – the DC resistance of the shield per unit length. For thick shield CBEM1 and CBEM2 give the same results, and ABEM produces values close to them (Fig. 5). For thin shield CBEM2 becomes very inaccurate (Fig. 6).

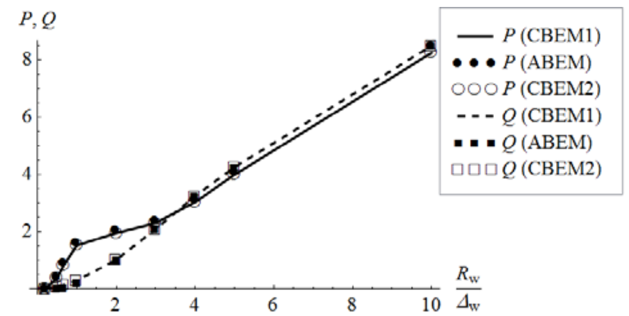

Fig. 5. Active and reactive power penetrating the shield versus R_w/Δ_w ($\Delta_w = \Delta_s$) for δ = 0.1

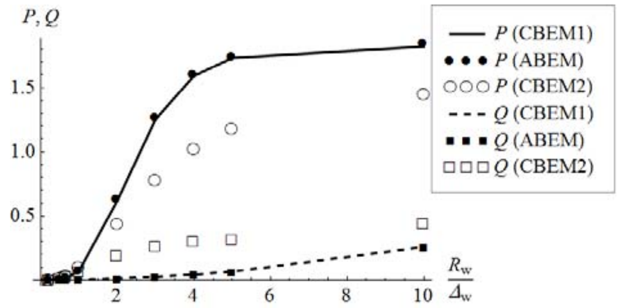

Fig. 6. Active and reactive power penetrating the shield versus R_w/Δ_w ($\Delta_w = \Delta_s$) for δ = 0.01

Concluding remarks

The presented computational model ABEM is a BEM-based model which uses an approximate expression to estimate the field inside thin shell instead of the standard BEM equation. In comparison to the conventional BEM, such an approach eliminates the presence of nearly singular integrals in the main matrix and leads to system of equations with smaller main matrix. This results in much shorter time of computation and memory demand while keeping the errors on acceptable levels.

Of course, it should be kept in mind that this approximation has its limitations. Numerical tests showed that ABEM works well for thin shells, and confirmed its usability in the considered types of problems. This is also confirmed by certain theoretical considerations not included in this paper due to brevity.

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