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Application of integral equations for the capacitance calculation in the hybrid microcircuit conductive paths system

Abstract. *The principles and results of capacitance calculations in the system of N conductors located in the a dielectric environment have been presented in the paper. The partial capacitances - self- and mutual - for such a system, considering the restricted and unrestricted areas have been determined. The subject of analysis was a system of N parallel, infinitely long conductive paths placed on a substrate of hybrid microcircuit. For the determination of partial capacitance the Fourier integral transformation method (presented in previous publications) has been applied.*

Streszczenie. *W artykule zaprezentowano zasady i rezultaty obliczania pojemności w systemie N przewodników umieszczonych w środowisku dielektrycznym. Określono pojemności cząstkowe własne i wzajemne dla takiego systemu, rozpatrując obszary ograniczone i nieograniczone. Przedmiotem analizy był układ równoległych, nieskończenie długich ścieżek przewodzących umieszczonych na podłożu mikroukładu hybrydowego. Do wyznaczenia pojemności cząstkowych zastosowano (prezentowaną we wcześniejszych publikacjach) metodę przekształceń całkowych Fouriera. (**Zastosowanie równań całkowych do obliczania pojemności w systemie ścieżek przewodzących mikroukładu hybrydowego).*

Keywords: hybrid microcircuit, partial capacitances, Fourier integral transformation, collocation method.
Słowa kluczowe: mikroukład hybrydowy, pojemności cząstkowe, przekształcenie całkowe Fouriera, metoda kolokacji

Introduction

Microelectronic hybrid circuit is a structure in which particular elements, active and passive, are electrically connected by means of conductive paths located, most often, on a dielectric substrate. Specificity of such circuit, conditioned by the manufacturing technology, miniaturization level, planarity of structure and related with it topography of conductive paths, can be cause of mutual couplings between the elements among which capacitive couplings are of great significance. This paper is a continuation of a set of publications concerning calculation of capacitance in microcircuits with various configuration of conductive paths [1] and the subject of discussions is a circuit consisting of N number of such layers. They are able to accumulate a charge and, directly, to accumulate energy. A unit of measure for such ability is the capacitance value of the path system. When looking for interrelations between the charges and potentials of particular path one should consider the fact that the charges depend both on the potential of a given conductor, as well as distribution and potential of all other. For that reason so-called self partial capacitance or mutual capacitance values between each conductive layer and all other ones are considered in the circuit. To determine capacitance it is enough to determine the potential distribution satisfying the Laplace's equation.

Capacitance determining in the circuit of conductors

If $\varphi_1, \varphi_2,\ldots, \varphi_N$, potentials are applied to each of N conductors located in the dielectric environment of ε permeability than $Q_1, Q_2\ldots, Q_N$ electrical charges will be induced on $S_1, S_2\ldots, S_N$ surfaces of this conductors and they will create an electric field of $\boldsymbol{E}$ intensity. The vector can be expressed by φ, potential. The φ function is harmonic and on each S_k surface of k conductor (k=1,…, N) has φ_k value. This - among many possible ways - can be presented in the following form:

$$\varphi(x,y,z) = \sum_{i=1}^{N} \varphi_i \cdot \xi_i(x,y,z) \tag{1}$$

where each of the ξ_i functions is harmonic and on the i conductor it equals 1 while on other ones it equals zero. On each k of S_k surface the electric charge Q_k can be defined by the following relation:

$$Q_k = -\varepsilon \iint_{S_k} \operatorname{grad}\varphi \cdot \boldsymbol{n}_k \cdot dS_k \tag{2}$$

where $\boldsymbol{n}_k$ is a unit vector perpendicular to S_k surface. Substituting (1) to (2) it can be obtained:

$$Q_k = \sum_{i=1}^{N} \varphi_i \cdot C_{ki} \tag{3}$$

where

$$C_{ki} = -\varepsilon \iint_{S_k} \frac{\partial \xi_i}{\partial n_k} dS_k \tag{4}$$

C_{ki} factors defined by the dependency (4) are called capacitance factors if $k = i$ and induction factors if $k \neq i$ [2]. In other literature sources [3,4,5] self partial capacitance terms are used (for $k = i$) and mutual ones (for $k \neq i$). Mutual partial capacitance values are characterised by a significant property, i.e.:

$$C_{ki} = C_{ik} \tag{5}$$

The ξ_k functions do not depend on φ_k potentials values, but only on the circuit's geometry. In this meaning they are of universal nature; they make it possible to solve the Laplace's equation at any φ_k values. However, one needs to bear in mind the fact that the aforementioned functions have different properties in limited systems and different in unlimited ones. The first of them refers to the internal Drichlet's condition for solution of the Laplace's equation and the other one - to the external condition (Fig. 1).

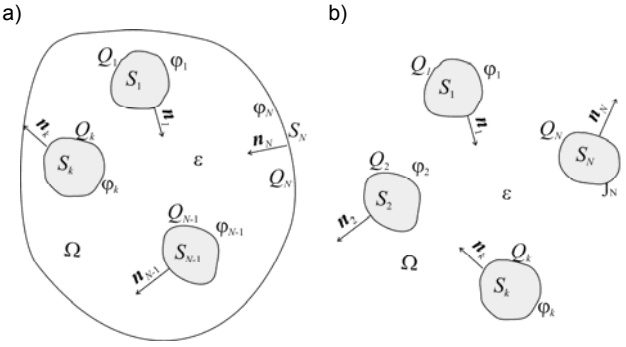

Fig.1.System of N conductors, region Ω: a) limited, b) unlimited

If, in a limited systems (Fig. 1.1a) one assumes that the values of potentials are identical: $\varphi_1=\varphi_2=\ldots=\varphi_N=\varphi_0$ the function $\varphi(x,y,z)$ defined in the equation (1) is constant in the entire

region Ω. Thus, on the basis of dependency (1) and (4) it can be described:

$$C_{kN} = -\sum_{i=1}^{N-1} C_{ki} \tag{6}$$

where k assumes values from 1 to N.
In order to determine the capacitance values the following system of equations is formed:

$$\begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{bmatrix} = [C] \cdot [\varphi] \tag{7}$$

where [C] matrix is called the *generalized capacitance matrix* [4]. For a limited systems of conductors, taking into account the dependency (7), partial capacitance values can be presented in the following way:

$$\begin{cases} C_{1N} = -(C_{12} + C_{13} + \ldots + C_{1N-1}) = C_{N1} \\ C_{2N} = -(C_{22} + C_{23} + \ldots + C_{2N-1}) = C_{N2} \\ \vdots \\ C_{NN} = -(C_{N2} + C_{N3} + \ldots + C_{NN-1}) \end{cases} \tag{8}$$

a and after consecutive transformations it can be obtained:

$$\sum_{k=1}^{N} Q_k = (\varphi_1 - \varphi_N)\sum_{k=1}^{N} C_{k1} + \ldots + (\varphi_{N-1} - \varphi_N)\sum_{k=1}^{N} C_{k(N-1)} \tag{9}$$

and, since each of the partial capacitance sums present at the right side of the equation (9) is - when taking into account (5) and (6) - equals zero, thus, for conductors in a limited area:

$$Q_1 + Q_2 + \ldots + Q_N = 0 \tag{10}$$

This property is characteristic only for limited systems; it is not present in unlimited areas, because in that case a different solution of the Laplace's equation exists and it is not constant, even if its border values are constant.

Especially interesting case is the one for N = 2. This is what results, adequately, from the system of equations (7) and dependency (4):

$$\begin{cases} Q_1 = \varphi_1 C_{11} + \varphi_2 C_{12} \\ Q_2 = \varphi_1 C_{21} + \varphi_2 C_{22} \end{cases} \tag{11}$$

$$\begin{cases} C_{11} = -\varepsilon \iint_{S_1} \dfrac{\partial \xi_1}{\partial n_1} dS_1 \\ C_{12} = -\varepsilon \iint_{S_1} \dfrac{\partial \xi_2}{\partial n_1} dS_1 = -\varepsilon \iint_{S_1} \dfrac{\partial (1-\xi_1)}{\partial n_1} dS_1 = -C_{11} \\ C_{22} = -C_{21} \end{cases} \tag{12}$$

As a result the $Q_1 = (\varphi_1 - \varphi_2)C_{11}$ or $Q_2 = (\varphi_2 - \varphi_1)C_{22}$ are determined, and thus a classic definition of capacitance: (13)

$$C = C_{11} = C_{22} = \left| \frac{Q_1}{\varphi_1 - \varphi_2} \right| = \left| \frac{Q_2}{\varphi_2 - \varphi_1} \right|$$

In the case of unlimited systems and number of conduits N = 2 the dependency (13) is not always met. This results from the fact that the condition Q_1 = -Q_2 can be met not at any φ_1 and φ_2 values. To determine capacitance in such circuit, taking advantage of equation (13), it is assumed that Q_1 = -Q_2. Than the dependency between φ_1 and φ_2 potentials is:

$$\varphi_2 = -\frac{C_{11} + C_{21}}{C_{22} + C_{21}} \varphi_1 \tag{14}$$

After substituting the dependency (14) to the system of equations (11) and simple transformations, it can be expressed:

$$\begin{cases} \varphi_1 = \dfrac{C_{22} + C_{21}}{C_{22}C_{11} - C_{21}C_{12}} Q \\ \varphi_2 = -\dfrac{C_{11} + C_{12}}{C_{22}C_{11} - C_{21}C_{12}} Q \end{cases} \tag{15}$$

thus, taking advantage of (13) and assuming that $C_{11}C_{22}$ - $C_{12}C_{21} \neq 0$ C capacitance is calculated:

$$C = \left| \frac{Q}{\varphi_1 - \varphi_2} \right| = \left| \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{22} + C_{12} + C_{21}} \right| \tag{16}$$

Planar structures being the subject of considerations consist of a systems of paths configurations where parasitic capacitances are observed. In real electronic circuits the potential of particular pathways is determined with relation to so-called reference conductor (also called a zero, mass or reference conductor). Bearing in mind this fact and assuming that these types of structures belong to unlimited systems (Fig. 1.1b) one can determine their partial capacitance values taking into account the reference conductor [4,5] Selecting N path as the reference conductor the U_i voltage is defined as the potential difference between the i path and the reference conductor N:

$$U_i = \varphi_i - \varphi_N, \qquad i = 1, 2, \ldots, N\text{-}1 \tag{17}$$

and a new capacitance matrix [C^*] is formed. Its dimensions (N-1 x N-1) take into account the reference conductor.

$$\begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{N-1} \end{bmatrix} = \begin{bmatrix} C^*_{11} & C^*_{12} & \cdots & C^*_{1N-1} \\ C^*_{21} & C^*_{22} & \cdots & C^*_{2N-1} \\ \vdots & \vdots & \ddots & \vdots \\ C^*_{N-11} & C^*_{N-12} & \cdots & C^*_{N-1N-1} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-1} \end{bmatrix} \tag{18}$$

In order to calculate partial capacitance values C^*_{ij} (i, j = 1...N-1), the potential $\varphi_i = U_i + \varphi_N$ determined from relation (17) should be substituted to equations (7), as a result of what is obtained:

$$\begin{cases} Q_1 = C_{11}(U_1 + \varphi_N) + \ldots + C_{1(N-1)}(U_{N-1} + \varphi_N) + C_{1N}\varphi_N \\ Q_2 = C_{21}(U_1 + \varphi_N) + \ldots + C_{2(N-1)}(U_{N-1} + \varphi_N) + C_{2N}\varphi_N \\ \vdots \\ Q_N = C_{N1}(U_1 + \varphi_N) + \ldots + C_{N(N-1)}(U_{N-1} + \varphi_N) + C_{NN}\varphi_N \end{cases} \tag{19}$$

Assuming that condition (10) is met in the unlimited system, after dispensing φ_N potential in the equations (19) and simple arithmetic operations the following formula is obtained, which enables to determine the partial capacitance values in the [C^*] matrix:

$$C^*_{ij} = C_{ij} - \frac{\left(\sum_{n=1}^{N} C_{in}\right)\left(\sum_{k=1}^{N} C_{kj}\right)}{\sum_{j=1}^{N}\sum_{i=1}^{N} C_{ij}} \tag{20}$$

It is worth noticing that the first summing up in the numerator of equation (20) is the sum of all *i* elements of the line, while the second one - the sum of all elements of *j* column of C matrix and, in the denominator, there is the sum of all elements of the matrix [5].

Assuming $N = 2$ number of conductors in an unlimited system and selecting one of them as the reference conductor and then, using the procedure described above, a dependency to C^* capacitance identical with the formula (16) is obtained.

Mathematical model

The subject of the analysis is a circuit consisting of N number of parallel, infinitely long conductive layers (paths) located at one side of the microcircuit substrate of h thickness and infinite length and width (Fig. 2).

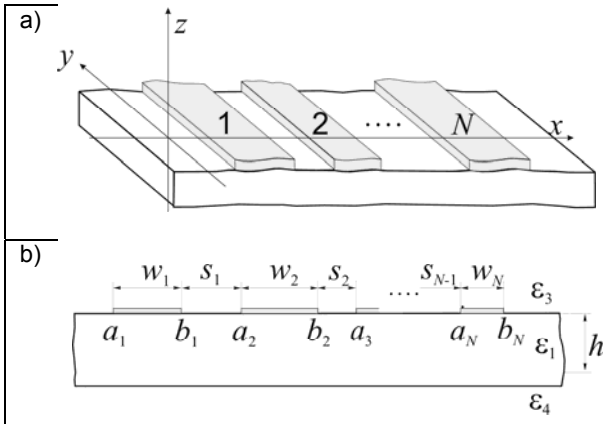

Fig.2. Distribution of conductive paths on the one side of the microcircuit substrate: a) general view, b) cross section z-x

Determining partial conductance values (calculated per unit of length) required, first of all, determination that would meet the Laplace's equation - distribution of potential. This two-dimensional boundary value problem was solved using the Fourier integral transform method (FIT). The following simplifying assumptions were implemented:

- the conductive path thickness is negligible;
- the width of each path is $(b_i\text{-}a_i)=w_i$, $i=1, \ldots, N$
- the dielectric permittivity of the substrate is ε_1, the area above its surface - ε_3, and below substrate surface - ε_4;
- the conductive layers were supplied potential φ_i and $\lim \varphi = 0$ $x^2 + z^2 \to \infty$

Potential $\varphi(x,z)$ can be expressed as:

$$\varphi(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha,z)\cdot e^{j\alpha x} d\alpha \tag{21}$$

where

$$F(\alpha,z) = \int_{-\infty}^{\infty} \varphi(x,z)\cdot e^{-j\alpha x} dx \tag{22}$$

is the transformation of $\varphi(x,z)$. The solution can be described as:

$$F = \begin{cases} R_1 e^{-|\alpha|(z-h)} & z \ge h \\ R_2 \sinh(\alpha z) + R_3 \cosh(\alpha z) & h \ge z \ge 0 \\ R_4 e^{|\alpha|(z+h)} & z \le -h \end{cases} \tag{23}$$

$R_1(\alpha) - R_4(\alpha)$ coefficients are derived from classical electrostatic field conditions:

$$\varphi(x,z)\big|_{z=0^+} = \varphi(x,z)\big|_{z=0^-} \tag{24}$$

$$\varphi(x,z)\big|_{z=-h^+} = \varphi(x,z)\big|_{z=-h^-} \tag{25}$$

$$D(x,z)\big|_{z=-h^+} = D(x,z)\big|_{z=-h^-} \tag{26}$$

$$D_n(x,z)\big|_{z=0^+} - D_n(x,z)\big|_{z=0^-} = \begin{cases} q_i(x) & a_i \le x \le b_i \\ 0 & x \notin (b_i - a_i) \end{cases} \tag{27}$$

As a result a system of integral equations determining the value of potentials obtained:

$$\varphi_i(x,0) = \frac{1}{\pi\varepsilon_3}\sum_{i=1}^{N}\int_{a_i}^{b_i} q_i(v) H(v,x)\, dv \tag{28}$$

where $H=H_r+H_o$ is a kernel of integral and includes regular part H_r und irregular – H_o:

$$H_r = \frac{\varepsilon_w(1-\varepsilon_d{}^2)}{\varepsilon_w+\varepsilon_d}\int_0^{\infty} G(\alpha)\cdot\frac{\cos(\alpha(x-v)}{\alpha} d\alpha$$

$$G(\alpha) = \frac{\operatorname{th}(\alpha h)}{\varepsilon_w + \varepsilon_d + (\varepsilon_d\varepsilon_w + 1)\operatorname{th}(\alpha h))}$$

$$H_o = \frac{\varepsilon_d}{\varepsilon_w+\varepsilon_d}\ln\frac{1}{|x-v|},$$

a $\varepsilon_d=\varepsilon_1/\varepsilon_4$, $\varepsilon_w=\varepsilon_1/\varepsilon_3$.

The equation system (28) was solved by a numerical calculation method, using the collocation method [6] and performing the discretisation of equations (Fig.3). It was also assumed, that equation system of distribution of electrical charge density q is fulfilled in points: $x_{m(M_i)}=a_i+(m\text{-}0.5)d_i$ for $m=1, \ldots, M_i$, and $q_{i(M_i)\,k(M_j)}$=const. for $v_k<v<v_{k+1}$, $v_{k(M_j)}=a_j+(k\text{-}1)d_j$, $d_j=(b_j-a_j)/M_j$, for $k=1, \ldots, M_j$; i and $j=1, \ldots, N$ denote the number of conductive paths.

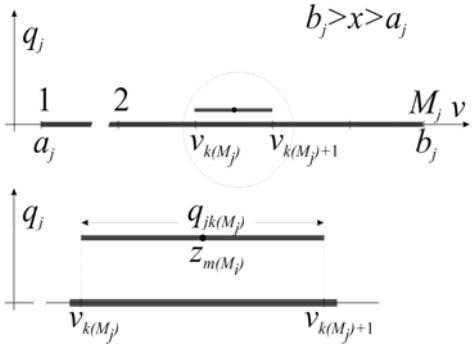

Fig. 3. Partition of layer widths; M_i - number of segments

The equations system (28) and can be now expressed as:

$$\begin{aligned} &\sum_{k=1}^{M_1} q_{1k} W11 + \sum_{k=1}^{M_2} q_{2k} W12 + \cdots + \sum_{k=1}^{M_N} q_{Nk} W1N = \varphi_1 \\ &\sum_{k=1}^{M_1} q_{1k} W21 + \sum_{k=1}^{M_2} q_{2k} W22 + \cdots + \sum_{k=1}^{M_N} q_{Nk} W2N = \varphi_2 \\ &\vdots \qquad\qquad \vdots \qquad\quad \ddots \quad \vdots \qquad = \quad \vdots \\ &\sum_{k=1}^{M_1} q_{1k} WN1 + \sum_{k=1}^{M_2} q_{2k} WN2 + \cdots + \sum_{k=1}^{M_N} q_{Nk} WNN = \varphi_N \end{aligned} \tag{29}$$

in which

$$Wij_{m(M_i),k(M_j)} =$$

(30)
$$= \frac{\varepsilon_d}{\varepsilon_w + \varepsilon_d} Iij_{m(M_i),k(M_j)} + \frac{\varepsilon_w(1-\varepsilon_d{}^2)}{\varepsilon_w + \varepsilon_d} Fij_{m(M_i),k(M_j)}$$

are called the coefficients of potential.

$$Iij_{m(M_i),k(M_j)} = \int_{v_k}^{v_{k+1}} \ln \frac{1}{|v - x_m|} dv$$

$$Fij_{m(M_i),k(M_j)} = \int_0^{\infty} \frac{G(\alpha)}{\alpha} \left[\int_{v_k}^{v_{k+1}} \cos(\alpha(v - x_m)) dv \right] d\alpha$$

This algebraic equation system (29), can be presented in matrix form:

(31) $$[\boldsymbol{\varphi}]=[\boldsymbol{W}]\cdot[\boldsymbol{C}]$$

whence the matrix of partial capacitance is

(32) $$[\boldsymbol{C}]=[\boldsymbol{W^{-1}}]$$

Numerical calculations and experimental verification

Based on presented mathematical model suitable numerical procedures were created (basing on algorithms used in previous studies [1]), for realization of which a professional Mathcad 13 program was used. In these procedures one assumed division of the pathways' width to 200 segments and, also, the upper limit for numerically calculated improper integrals F_{ij} was determined to approx. $5 \cdot 10^3$.

On this basis a series of simulations were carried out, changing the number of conductive paths, geometrical parameters of the system, as well as the number of the reference path. Exemplary results are presented for a circuit of 3 paths of configuration as in figure 2b; the third path was assumed as the reference conductor. The calculations were preformed assuming the following circuit parameters: $w_1 \div w_3$=0,6 mm, $s_1=s_2$=0,4 mm, $\varepsilon_3=\varepsilon_4$=1, ε_1=10,2.Results of the calculations were partially verified experimentally, by comparison with measurement results in real circuits. For this purpose the suitable test circuits were designed and performed. Those circuits (10 test samples) have been made on substrates of 100x65x0,635 mm using the alundum ceramics 96% Al_2O_3 with dielectric constant ε_1=10,2. To perform the measurements RLC bridge by Agilent 4282A was used. In order to eliminate the impact of systematic errors to the result of the calculations, the measurement system - when thermal conditions became stable - was subject to a process of automatic OSL *(Open, Short, Load)* correction. The values obtained when taking into account the reference conductor and results of the simulations presented in a form of [Cs^*] matrix and measurements - [Cm^*] (in pF/m), can be considered as acceptable.

$$Cs^* = \begin{bmatrix} 77{,}518 & -63{,}11 \\ -63{,}11 & 126{,}21 \end{bmatrix} \quad Cm^* = \begin{bmatrix} 85{,}22 & -65{,}38 \\ -65{,}38 & 126{,}82 \end{bmatrix}$$

The correctness of the elaborated calculation method can also be confirmed by the distribution of the charge density in the area of conductive paths. This density is practically constant on the most part of their areas and suddenly increases near the edges (Fig.5).

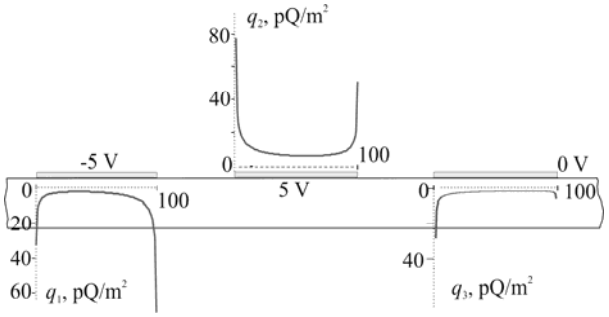

Fig.5. Results of numerical calculation of charge distribution

Correctness of calculations for the program were additionally checked by comparing them with results obtained when using other calculation methods [4]. For circuit parameters of $w_1=w_2=w_3$=0,381 mm, $s_1=s_2$=1,143 mm, h=1.1938 mm, ε_1=4.7 (glass epoxy) was obtained:

- FIT method $$C=\begin{pmatrix} 29.7244 & -20.2901 \\ -20.2901 & 40.5803 \end{pmatrix}$$

- point-matching method $$C=\begin{pmatrix} 29.7386 & -20.2999 \\ -20.2999 & 40.5997 \end{pmatrix}$$

- Galerkin method $$C=\begin{pmatrix} 29.7632 & -20.3140 \\ -20.3140 & 40.6208 \end{pmatrix}$$

and this fully proves correctness of the method.

Conclusion

The procedures and program for calculating per-unit-length capacitance values for a conducting path systems enable analysis of geometrical and physical factors which have an impact on the capacitance value. They also give a possibility - when using the same algorithms - for performing calculations for circuits of a different configuration (with a screen plane, multi-layer, etc.). Satisfactory conformity of the calculation results and measurements of capacitance values, as well as good convergence of numerical simulations of Fourier's integral transformation method, when compared to calculation results during which other methods were used (Galerkin's, point matching), confirms correctness of the elaborated procedures.

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