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Skin-effect in Conductor of Rectangular Cross-section — Approximate Solution

Abstract. Current distribution in massive conductor of rectangular cross section due to the skin-effect can be calculated approximately by two simple formulae. One was found in the literature, the second, based on constant surface current, was derived recently. In the paper the general approach based on iterative technique was derived for more precise calculations. Since some unusual features follow from the formulae, the experimental verification is necessary. As the current density cannot be measured, magnetic field of the conductor is calculated by the combination of analytical formula and numeric integration. Results exhibit acceptable agreement between approximation theory based on constant surface current and experiment. More accurate approximation requires more precise experiment.

*Streszczenie. Analizowano rozkład prądu w przewodniku o przekroju prostoktątnym z uwzględnieniem efektu naskórkowości. Analizowano dwie metody - jedną znaną z literatury, drugą bazującą na stałym prądzie powierzchniowym. Ponieważ nie był możliwy bezpośredni pomiar gęstości prądu analizowano pole magnetyczne. (**Efekt naskórkowości w przewodzie prostokątnym - analiza aproksymacyjna**)*

Keywords: eddy currents, 2D skin-effect, massive conductor, iterative technique, magnetic field measurement, 2D skin-effect verification
Słowa kluczowe: prądy wirowe, efekt naskórkowości, rozkład gęstości prądu

Introduction

Well-known skin-effect, which leads to nonuniform current density distribution in conductors, exists in principle at all frequencies. However, at low frequency the uniform current distribution is adequate and at high frequencies the current flowing in a very thin surface layer is a good approximation. In the medium range of frequencies the nonuniform current density must be considered. Typical applications are in modern distribution points. Because of high currents, distribution points use conductors of large cross section and the current density is not uniform in them even at low technical frequencies.

The skin-effect is given by differential equations. Analytical solution is known only in very simple cases as half space, infinite layer or infinite conductor of circular cross-section. There are presented in every text book, for instance [1]. In the case of conductor rectangular cross-section a relatively simple analytical solution cannot be found, probably. Two ways exist: application of finite element method (FEM) or approximate analytical solution. While the FEM solution is in every commercial system, only one such approximation was found in literature [2]. Another, more complex, approximation was derived by authors [3].

The paper presents the general approach that can be used to get an approximative solution of selected accuracy. Since the only simple skin-effect verification is measurement of the magnetic field outside the conductor, the paper shows the comparison of both the known approximate solutions with experimental results.

Theory

We consider infinite long conductor of rectangular cross-section. Both the geometry and coordinate system are shown in Fig. 1. We suppose current of volume density $\vec{i}$ flowing along Z axis. Since the electric field strength $\vec{E}$ is proportional current density, $\vec{E} = \gamma \vec{i}$, we formulate the basic equations for electric field strength $\vec{E}$ and magnetic flux density $\vec{B}$.

Electric field strength has only z-component E_z, $\vec{E} = (0, 0, E_z)$, which under the harmonic excitation and in the steady state satisfies the partial differential equation

$$(1) \qquad \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - k^2 E_z = 0,$$
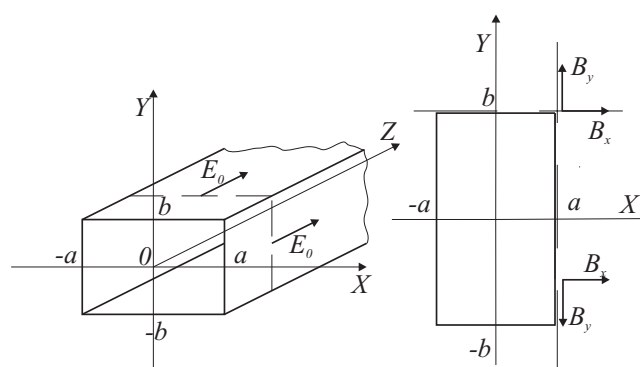

Fig. 1. Conductor geometry, coordinate system and electromagnetic field components.

where the complex constant k is given by the formula

$$(2) \qquad k = \sqrt{\mathrm{j}\omega\mu\gamma} = (1+\mathrm{j})\sqrt{\frac{\omega\mu\gamma}{2}},$$

where $\omega = 2\pi f$ is the angular frequency (f is frequency), μ is the permeability and γ is conductivity.

We use complex symbolic calculus because of its simplicity and effectiveness. Then j is the imaginary unit with definition $\mathrm{j}^2 = -1$. Almost all the considered quantities are complex ones. Therefore, we do not use special symbols to highlight this feature. Physical meaning have their amplitude and phase, or real and imaginary parts.

The general solution of equation (1) is a product of hyperbolic functions. Because of physical reasons, the only applicable are hyperbolic cosines

$$(3) \qquad E_z(x, y) = E_{z0} \cosh(k_x x)\cosh(k_y y),$$

where k_x and k_y are complex wave constants.

After the substitution of solution (3) into equation (1) we get the condition for unknown wave constants

$$(4) \qquad k_x^2 + k_y^2 = k^2,$$

where constant k is given by equation (2). There is infinite number of solutions (3), since one wave constant, say k_x, can be selected independently. According to equation (4) wave constants k_x and k_y lie on the circle of radius k, see Fig. 2.

Magnetic field is determined from electric one using the Maxwell equations. Optimum for our purposes is the Fara-
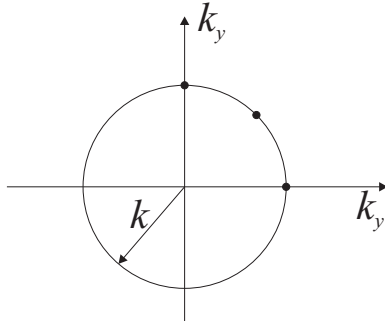
Fig. 2. Wave constant determination. Used choices are denoted by circles.

day's Law that leads to the equation

$$(5) \qquad \vec{B} = \frac{1}{j\omega} \operatorname{rot} \vec{E}.$$

After substitution of solution (3) into these equations (5) we get formulae for components of the magnetic flux density

$$
\begin{aligned}
(6) \qquad & B_x(x, y) = B_{x0} \sinh(k_x x) \cosh(k_y y), \\
& B_y(x, y) = B_{y0} \cosh(k_x x) \sinh(k_y y), \\
& B_z(x, y) = 0,
\end{aligned}
$$

where constants B_{x0} and B_{y0} are determined by previous ones E_{z0}, k_x and k_y. The solution (6) respects physical conditions and geometrical symmetry for magnetic flux density.

General solution is the linear combination of partial equations for selected values of wave constant k_x, for instance. Therefore, it is possible to control the degree of approximation. The coefficients must be determined from boundary conditions. Unfortunately, they are unknown. The probably consequence of this difficulty is that the problem is not presented in the literature.

Special solutions

Only two special solutions are known to us. Just the product in (3) was used in Ref. [2]. The approximate solution has the simplest form

$$(7) \qquad E(x, y) = C k_x k_y \frac{\cosh(k_x x) \cosh(k_y y)}{\cosh(k_x a) \cosh(k_y b)},$$

where the wave constant k_x and k_y are given by formulae

$$(8) \qquad k_x = k \sqrt{\frac{b}{a+b}}, \qquad k_y = k \sqrt{\frac{a}{a+b}},$$

where $2a$ and $2b$ are width and height of conductor, see Fig. 1. Constant C in (7) is determined from the total current I. In the wave constant circle in Fig. 2 the point given by wave constants in formulae (8) is somewhere in the first quadrant.

More general solution [3] considers three terms with special wave numbers: $(k_x = 0, k_y = k)$, $(k_x = k, k_y = 0)$ and $k_x = k_y = k/\sqrt{2}$. They are shown in Fig. 2 by small solid circles. The points have privileged positions in the first quadrant.

In order to get simple analytical solution, two simplifications were used: The surface current density on the conductor is constant and the third (product) term has wave numbers $k_x = k_y = k$. Then the solution has the analytical form

$$
\begin{aligned}
(9) \qquad E(x, y) = & -E_o \frac{\cosh(kx) \cosh(ky)}{\cosh(ka) \cosh(kb)} + \\
& + E_o \frac{\cosh(kx)}{2 \cosh(ka)} + E_o \frac{\cosh(ky)}{2 \cosh(kb)},
\end{aligned}
$$

where E_0 is the constant electric field strength on the conductor surface that is determined from the total current I.

General solution

In general we can select several wave constants as suitable points in the first quadrant of wave number circle in Fig. 2. The use of the first quadrant only is not a limitation, since the wave constants from other quadrants differ only in the sign. Since cosh functions in partial solution (3) are even ones, the sign has no effect. Therefore, the general approach is to take several suitable points in the first quadrant, calculate their wave numbers and substitute them into partial solution (3). The analytical solution of skin-effect will be the sum of partial ones with unknown coefficients. The task is to determine the coefficients. Usually the boundary conditions are used for this purpose, but in our case the value of electric field strength, for instance on the wire surface, is unknown. There are two basic possibilities: the use of FEM or the iterative approach.

The simplest possibility is to find exact values in special points. In principle, we can use numerical results of thoroughly applied FEM. However, FEM points differs from application to application and the accuracy may not be satisfactory.

The iterative approach appear to be a more suitable one. Electric and magnetic fields are coupled. As the Faraday's Law (5) determines the magnetic field from the electric one in the conductor, then the Amper's Law

$$(10) \qquad \vec{E} = \frac{1}{\mu\gamma} \operatorname{rot} \vec{B},$$

has an opposite function. The formula (10) calculates the electric field from the magnetic one. If the electric field calculated from Amper's Law (10) is identical with the original one, then we have got the exact solution.

Therefore the general approach uses the iteration technique:

1. First, we estimate the initial electric field strength $\vec{E}^{(1)}$ in the conductor. The simplest estimation, constant electric field strength, is not suitable. The optimum may be formula similar to (9) that can contain more products of correct values for wave constants k_x and k_y.

2. Magnetic field flux density $\vec{B}^{(1)}$ is calculated from electric field by the use of formula (5).

3. New electric field strength $\vec{E}^{(2)}$ is found using the formula (10) from the magnetic field obtained in the previous step.

4. If the difference between new and old electric field strengths is less than the prescribed value, the solution was found. Otherwise, we continue from the step 2 using the new electric field strength $\vec{E}^{(1)} = \vec{E}^{(2)}$.

If we use formula similar to (9) for the initial electric field strength, the coefficients in formula for magnetic flux density (5) can be found analytically. The expressions are not simple, however. Analogically, by the use of suitable substitution, the analytical formulae for coefficients of new electric field can be obtained. Therefore, numerical values of coefficients calculated in the program are exact.

The iterative procedure can be programmed relatively easily, since analytical formulae for coefficients can be derived. However, the formulae will not be simple. The basic output of the program are numerical values of coefficients in analytical formula that close approximates the electromagnetic field in the conductor.

Experiment and Results

Since the current distribution in the conductor cannot be measured, the only simple experimental verification of the skin-effect calculations is the external magnetic field. Both the magnetic field calculation by the use of Biot-Savart Law and the own experimental apparatus are described in detail in Ref. [3]. We have used massive three phase straight conductors for three phase net. The cross section has width of 5 mm and height of 40 mm. In order to eliminate eddy currents experimentally, only the central conductor was fed by current I_2. Opposite currents $I_1 \doteq I_3 \doteq I_2/2$ flow in lateral conductors. In order to eliminate remaining eddy effects, only extreme values of scanned waveforms of currents and magnetic field components are used. Details can be found in Ref. [4].
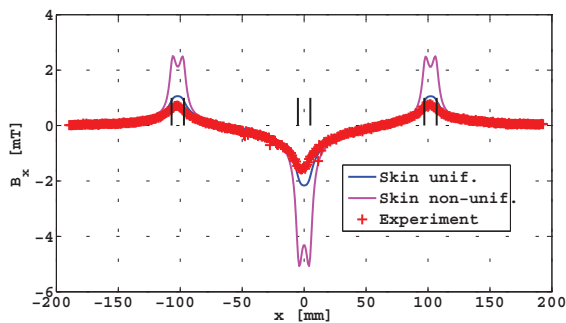

Fig. 3. Comparison of solutions — horizontal component of magnetic flux density.

The comparison of experiment and both the approximative theories are shown in Fig. 3 for the horizontal component of magnetic flux density B_x. The frequency of current excitation was 1200 Hz (apparatus maximum) and approximately 110 A flowed in the central conductor. Positions of conductors are also sketched in the Figure.

Theoretical calculations were made for both the approximations. Result for nonuniform surface current given by equations (7) is noted as "Skin nounif". The label "Skin unif" is used for application of uniform surface current approximation based on formulae (9). The labels therefore mean the current distribution on the conductor surface.
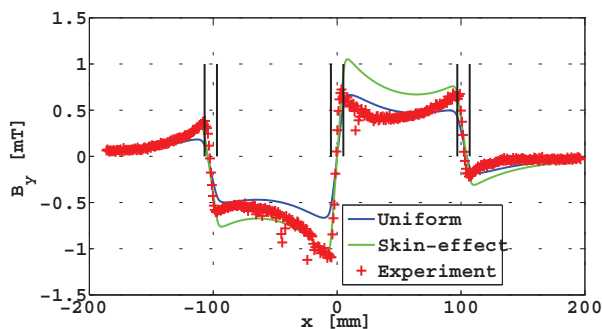

Fig. 4. Comparison of theory and experiment — vertical component of magnetic flux density.

Since the better agreement is for uniform surface current approximation, details for vertical magnetic flux density B_y are shown in Fig. 4. Theoretical curves are for uniform current distribution (DC case) and for skin-effect at frequency of 1200 Hz.

Discussion and Conclusion

Two approximate theories of skin-effect in massive conductor of rectangular cross section were mentioned. Theory of variable surface current exhibit worse agreement with experiment in comparison with theory of constant surface current, irrespective of the fact that exactly constant surface current is not physically possible. It can be explained by the fact that the constant surface current theory uses three terms and, therefore, has a higher degree of variability. It is the reason that a lot of calculations, for instance in Ref. [4] were made by the use of constant surface current model.

The proposed iterative method is a general approach. In principle we can achieve very high accuracy. On the other hand, FEM is very popular, not only in skin-effect calculations. Typical FEM disadvantages are: the values are not calculated in prescribed points, the accuracy is not known and values of other related quantities cannot be calculated simply. A big advantage of integrative method is analytical formula with several numerical coefficients that can be calculated relatively simply. Analytical formula allows easy calculation of related quantities as magnetic flux density, for instance.

More exact approximation requires more precise experiment. As it follows from Fig. 4 the experimental points are not anti-symmetrical. Due to the experimental errors, relative large deviations appear. Severe apparatus limitation is a low exciting current of about 100 A at frequency of 1200 Hz. At 45 Hz the current is more than 1000 A. Confirmation of corrected theory needs more accurate measurement of magnetic flux density, which is not a simple task. It was also the reason, why the iterative method was not developed at present time.

Both the experiment and calculations have shown that the skin-effect does not play an important role in examined configuration. At frequency of 45 Hz practically no difference was found between results for constant current density in the conductor and its approximative no-uniformity due to the skin-effect. It is not necessary to consider the Skin-effect at technical frequencies. From the practical point of view, eddy currents play more important role, but their inclusion is possible only by FEM, probably,.

REFERENCES
[1] Hanka L.: Theory of electromagnetic field, SNTL, Prague, 1975. (in Czech).
[2] Gerling, D.: Approximate analytical calculation of the skin effect in rectangular wires, International Conference on Electrical Machines and Systems, 2009. ICEMS 2009, pp. 1-6 15-18 Nov. 2009.
[3] Kosek M., Truhlar M., Richter A.: Magnetic Field of Massive Conductor at Low Frequency, Przeglad Eletryczny (Electrical Review), 87 (5), pp. 179-185, May 2011. ISSN 0033-2097.
[4] Truhlar M.: Nonlinear Distribution of Current Density in Three Phase Distribution Power Net, PhD Thesis, Technical University of Liberec, Liberec, Czech Republic, September 2011 (in Czech).

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