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Semi-analytical methods for optimal energy transfer in RC ladder networks

Abstract. *This paper presents a semi-analytical solution to the optimal control problem arising with energy transfer through transmission line. Distributed parameter RC line is approximated with a RC ladder network. Formulas for optimal control are determined up to a moment when matrix exponential has to be computed. This final element has to be computed numerically. Paper ends with results of simulations and a discussion of numerical issues.*

Streszczenie. *Praca przedstawia semi-analityczne rozwiązanie problemu sterowania optymalnego występującego przy przesyle energii poprzez linię długą. Linia długa RC jest aproksymowana za pomocą układu łańcuchowego typu RC. Wzory określające sterowanie optymalne są wyznaczone do momentu konieczności wyliczenia eksponenty macierzy. Ta część obliczeń musi być wykonana numerycznie. Praca kończy się wynikami symulacji i dyskusją zagadnień numerycznych.* ***(Metody semi-analityczne w optymalnym przesyle energii w układach drabinkowych RC)***

Keywords: energy transfer, RC ladder network, maximum principle, optimisation
Słowa kluczowe: przesył energii, układ drabinkowy RC, zasada maksimum, optymalizacja

Introduction

Analysis of electrical ladder networks is an important aspects of control theory and theoretical electrical engineering. Their importance comes from connection to the models of infinite dimensional processes (especially thermal), where they can be used as efficient and intuitive finite dimensional approximations. Authors' recent works considered especially stabilisation of networks, among the others undamped LC type [4, 6] and nonlinear RC type [23]. Also under consideration were problems of control in ladder networks were considered in [2] and [3] and applications in modeling of household buildings [9, 10]. This work is a continuation of results presented in [2, 5, 20].

Optimal control problems arise in many real life situations. One interesting problem is heating with an electric current (see for example [21]). We consider an electric network shown in the figure 1. The resistance of the voltage source R_1 and the output resistance R_H are given. Assuming that $i_w(0) = 0$ the energy producing heat on the output resistance is given by

$$E = \int_0^T y(t) i_w(t) \mathrm{d}t$$

It is desired to transfer the required energy through the transmission medium at the same time minimising the energy delivered to the network.

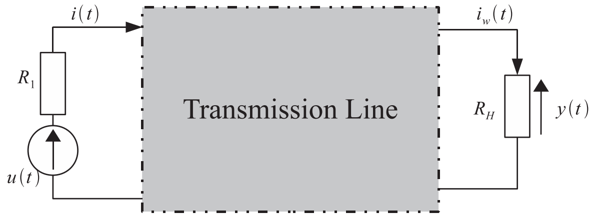

Fig. 1. Schematic representation of a transmission line

In order to find such input voltage $u(t)$ that would realise the desired goal one needs to formulate the mathematical model of the problem. Consider a homogeneous long electric RC transmission line, i.e. one where the parameters per the unit length (resistance r and capacity c) are constant and independent of the spatial variable z. An infinitesimal part of the transmission line is described by the equation

$$rc\frac{\partial x(t,z)}{\partial t} = \frac{\partial^2 x(t,z)}{\partial z^2}$$
$$t \geq 0$$
$$0 \leq z \leq l.$$

Let $z = ih$, $h = l/n$, $i = 0, 1, \ldots, n$ and $x(t, (2k-1)h/2) = x_k(t)$, $k = 1, 2, \ldots, n$. We have

$$\frac{\partial^2 x(t,z)}{\partial z^2} \approx \frac{1}{h}\left(\frac{x(t,z+h) - x(t,z)}{h} - \right.$$
$$\left. - \frac{x(t,z) - x(t,z-h)}{h}\right)$$

for $z = (2k-1)h/2$ and $k = 1, 2, \ldots, n$. Then the RC transmission line can be approximated by the RC ladder network shown in the figure 2, where $R = rl$ and $C = cl$ (see for example [8]).

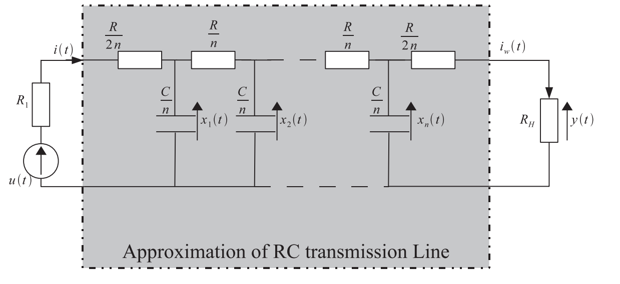

Fig. 2. Approximation of a RC transmission line with a RC ladder network

Optimal control problem

We consider the electric RC ladder network shown in the figure 2. Its parameters R, R_1, R_H and C are known. The system can be described by the following system of equations (see for example [2, 16, 18–20])

$$\begin{aligned} &\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ &\mathbf{x}(t) = [x_1(t) \quad x_2(t) \ldots x_n(t)]^{\mathsf{T}}, \\ &y(t) = \mathbf{W}\mathbf{x}(t) \end{aligned} \tag{1}$$

Where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a tridiagonal Jacobi matrix [16–18], such that

$$\mathbf{A} = [a_{ij}]$$

$$a_{ij} = \begin{cases} a_{ij} = 0, \text{ for } |i-j| > 1 \\ a_{ii} = -2\dfrac{n^2}{RC}, \text{ for } i = 2,\ 3,\ \ldots, n-1, \\ a_{11} = -(1+r(R_1))\dfrac{n^2}{RC}, \\ a_{nn} = -(1+r(R_H))\dfrac{n^2}{RC}, \\ a_{i+1,i} = \dfrac{n^2}{RC}, \text{ for } i = 1,\ 2,\ \ldots, n-1, \\ a_{i,i+1} = a_{i+1,i}, \text{ for } i = 1,\ 2,\ \ldots, n-1. \end{cases}$$

$$r(v) = \frac{2R}{2nv+R},$$
$$\mathbf{B} = \frac{n^2 r(R_1)}{RC}\mathbf{e}_1,$$
$$\mathbf{W} = \frac{n^2 r(R_H)R_H}{R}\mathbf{e}_n^{\mathsf{T}},$$

$$\mathbf{e}_1 = [1\quad 0\quad 0\quad \ldots\quad 0\quad 0]^{\mathsf{T}} \in \mathbb{R}^n,$$
$$\mathbf{e}_n = [0\quad 0\quad 0\quad \ldots\quad 0\quad 1]^{\mathsf{T}} \in \mathbb{R}^n.$$

Formulation of the problem

Let T and E be fixed, the goal is to find a control signal that in the time horizon T delivers an amount of energy E to the receiver R_H delivering minimal amount of the overall energy to the RC network (see for example [1]).

Formal definition of the problem is to find such control $u_0 \in U_d$ such that:

$$J(u) \geq J(u_0)$$
$$\forall u \in U_d,$$

where $J(u)$ is the performance index given by

(2) $$J(u) = \int_0^T u(t)i(t)\mathrm{d}t = = \frac{2n}{2nR_1+R}\int_0^T u(t)[u(t)-x_1(t)]\mathrm{d}t,$$

with $\mathbf{x}$ given by (1) and $\mathbf{x}(0) = 0$. U_d is the set of admissible controls

(3) $$U_d = \left\{u : \frac{1}{R_H}\int_0^T y(t)^2\mathrm{d}t = = \frac{n^2 R_H}{(nR_H+R/2)^2}\int_0^T x_n(t)^2\mathrm{d}t = E\right\}$$

It should be noted that the set $U_d \neq \emptyset$. To see that, examine for example $u(t) = \mathrm{const}$ such that:

$$\frac{1}{R_H}\int_0^T y(t)^2\mathrm{d}t = \frac{n^2 R_H}{(nR_H+R/2)^2}\int_0^T x_n(t)^2\mathrm{d}t = E.$$

Now, we consider the space $L^p(0,T)$, $p \in [1,\infty)$ with the norms $\|f\|_p = [\int_0^T |f(t)|^p\mathrm{d}t]^{1/p}$. From the Hölder inequality (see for example [15]) we have

$$\int_0^T u(t)x_1(t)\mathrm{d}t \leq \|ux_1\|_1 \leq \|u\|_2\|x_1\|_2.$$

The system (1) is asymptotically stable, controllable and observable (the pair $(\mathbf{A},\mathbf{B})$ is controllable [14] and $(\mathbf{W},\mathbf{A})$ is observable, see also [13]). Consequently:

$$(R_1 + R/2n)J(u) = \|u\|_2^2 - \int_0^T u(t)x_1(t)\mathrm{d}t \geq$$
$$\geq \|u\|_2^2 - \|u\|_2\|x_1\|_2 \geq -\|x_1\|_2^2/4$$

(see (2) and (3)),for every $u \in U_d$ the norm $\|x_1\|_2$ finite and $J(u) \to \infty$ as $\|u\| \to \infty$. Thus there exists the optimal control u_0, cf. (2). We can notice that $J(u) = J(-u)$.

Application of Maximum Principle

It is possible to construct an algorithm for determination of optimal control with use of the Maximum Principle [7, 22]. One needs to observe that the problem is an optimal control problem with equality constraints on terminal state. To do so new state variables are defined:

(4) $$\begin{aligned} &\dot{x}_{n+1}(t) = x_n(t)^2, \\ &x_{n+1}(0) = 0, \\ &\dot{x}_{n+2}(t) = u(t)(u(t)-x_1(t)), \\ &x_{n+2}(0) = 0, \end{aligned}$$

one can see that because $x_{n+1}(T)$ corresponds to the constraint given by the admissible set (3) and $x_{n+2}(T)$ is used to represent an integral performance index we have

$$x_{n+1}(T) = \frac{(nR_H+R/2)^2}{n^2R_H}E,$$
$$x_{n+2}(T) = \frac{2nR_1+R}{2n}J(u).$$

Let us introduce the adjoint function $\boldsymbol{\psi}\colon \mathbb{R} \to \mathbb{R}^n$ and introduce the following notation

$$\tilde{\mathbf{x}}(t) = [\mathbf{x}(t)^{\mathsf{T}}\quad x_{n+1}(t)\quad x_{n+2}(t)]^{\mathsf{T}}$$

and

$$\tilde{\boldsymbol{\psi}}(t) = [\boldsymbol{\psi}(t)^{\mathsf{T}}\quad \psi_{n+1}(t)\quad \psi_{n+2}(t)]^{\mathsf{T}}.$$

Then we obtain the Hamiltonian in the form:

(5) $$\begin{aligned} &H(\tilde{\boldsymbol{\psi}}(t),\tilde{\mathbf{x}}(t),u(t)) = \\ &\boldsymbol{\psi}(t)^{\mathsf{T}}[\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t)] + \psi_{n+1}(t)x_n(t)^2 + \\ &+ \psi_{n+2}(t)u(t)[u(t)-x_1(t)]. \end{aligned}$$

Using transversality conditions we have

$$\psi_{n+1}(t) = -\rho = \mathrm{const},$$

$$\psi_{n+2}(t) = -1,$$

$$\boldsymbol{\psi}(T) = 0 \in \mathbb{R}^n$$

and the adjoint function $\boldsymbol{\psi}$ is a solution to the following system of equations

(6) $$\dot{\boldsymbol{\psi}}(t) = -\mathbf{A}^{\mathsf{T}}\boldsymbol{\psi}(t) - \mathbf{e}_1u(t) - 2\rho\mathbf{e}_nx_n(t)$$

Using the Maximum Principle (see for example [7, 12, 22]), from (5) we get:

$$u(t) = \frac{1}{2}[\mathbf{B}^\mathsf{T}\boldsymbol{\psi}(t) + x_1(t)] \tag{7}$$

The control (7) depends on the real number ρ and is called the extremal control. We will search for the optimal control u_0 among the extremal controls (7).

From (1), (6) and (7), we obtain the canonical system in the following form:

$$\begin{gathered} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\boldsymbol{\psi}}(t) \end{bmatrix} = \mathbf{Z} \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\psi}(t) \end{bmatrix} \\ \mathbf{x}(0) = 0 \\ \boldsymbol{\psi}(T) = 0, \end{gathered} \tag{8}$$

where

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A} + \frac{1}{2}\mathbf{B}\mathbf{e}_1 & \frac{1}{2}\mathbf{B}\mathbf{B}^\mathsf{T} \\ -\frac{1}{2}\mathbf{e}_1\mathbf{e}_1^\mathsf{T} - 2\rho\mathbf{e}_n\mathbf{e}_n^\mathsf{T} & -\mathbf{A} - \frac{1}{2}\mathbf{e}_1\mathbf{B}^\mathsf{T} \end{bmatrix}, \tag{9}$$

$$\mathrm{e}^{\mathbf{Z}t} = \begin{bmatrix} \boldsymbol{\Phi}_1(t) & \boldsymbol{\Phi}_2(t) \\ \boldsymbol{\Phi}_3(t) & \boldsymbol{\Phi}_4(t) \end{bmatrix} \tag{10}$$

Then from (8) and (10) we have

$$\begin{gathered} \mathbf{x}(t) = \boldsymbol{\Phi}_2(t)\boldsymbol{\psi}(0) \\ \boldsymbol{\psi}(t) = \boldsymbol{\Phi}_4(t)\boldsymbol{\psi}(0). \end{gathered} \tag{11}$$

If $E \neq 0$, then $\mathbf{x}(t) \neq 0$, cf. (3). Thus from (11) we get $\boldsymbol{\psi}(0) \neq 0$. Since $\boldsymbol{\psi}(T) = 0$, cf. (8), from (11) we have:

$$\det \boldsymbol{\Phi}_4(T) = 0. \tag{12}$$

Obtaining the optimal control

The following algorithm can be used for the determination of optimal control. It is a variant of algorithm proposed in [20].

1. Determine the parameter ρ using equation (12).
2. Because $\mathrm{rank}\boldsymbol{\Phi}_4(T) = n - 1$ using gaussian elimination determine the dependence of solution of $0 = \boldsymbol{\Phi}_4(T)\boldsymbol{\psi}(0)$ on a single parameter γ in particular

$$\boldsymbol{\psi}(0) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ 1 \end{bmatrix} \gamma = \boldsymbol{\alpha}\gamma \tag{13}$$

3. From (3), (11) and (13) calculate γ and subsequently $\boldsymbol{\psi}(0)$. This will be discussed below.
4. From (7) and (11) determine

$$\begin{aligned} u(t) &= \frac{1}{2}[\mathbf{B}^\mathsf{T}\boldsymbol{\psi}(t) + x_1(t)] = \\ &= \frac{1}{2}\left[\mathbf{B}^\mathsf{T}\boldsymbol{\Phi}_4(t) + \mathbf{e}_1^\mathsf{T}\boldsymbol{\Phi}_2(t)\right]\boldsymbol{\psi}(0), \end{aligned} \tag{14}$$

As one can easilly see steps 1 and 2 determine the extremal solutions, while the optimal one is chosen in step 3 by fulfilling the equality constraint (3). Step 3 requires special discussion. Let us consider the integral in (3).

$$\int_0^T x_n(t)^2 \mathrm{d}t =$$

$$= \int_0^T \begin{bmatrix} \mathbf{x}(0) & \boldsymbol{\psi}(0) \end{bmatrix} \mathrm{e}^{\mathbf{Z}^\mathsf{T}t} \begin{bmatrix} \mathbf{e}_n\mathbf{e}_n^\mathsf{T} & 0 \\ 0 & 0 \end{bmatrix} \cdot$$

$$\mathrm{e}^{\mathbf{Z}t} \begin{bmatrix} \mathbf{x}(0) \\ \boldsymbol{\psi}(0) \end{bmatrix} \mathrm{d}t = \begin{bmatrix} \mathbf{x}(0) & \boldsymbol{\psi}(0) \end{bmatrix} \cdot$$

$$\cdot \int_0^T \mathrm{e}^{\mathbf{Z}^\mathsf{T}t} \begin{bmatrix} \mathbf{e}_n\mathbf{e}_n^\mathsf{T} & 0 \\ 0 & 0 \end{bmatrix} \mathrm{e}^{\mathbf{Z}t}\mathrm{d}t \begin{bmatrix} \mathbf{x}(0) \\ \boldsymbol{\psi}(0) \end{bmatrix} \mathrm{d}t$$

let us denote

$$\mathbf{Q} = \begin{bmatrix} \mathbf{e}_n\mathbf{e}_n^\mathsf{T} & 0 \\ 0 & 0 \end{bmatrix}.$$

We reformulate the integral

$$\int_0^T \mathrm{e}^{\mathbf{Z}^\mathsf{T}t}\mathbf{Q}\mathrm{e}^{\mathbf{Z}t}\mathrm{d}t =$$

$$= \mathrm{e}^{\mathbf{Z}^\mathsf{T}T}\mathrm{e}^{-\mathbf{Z}^\mathsf{T}T} \int_0^T \mathrm{e}^{\mathbf{Z}^\mathsf{T}t}\mathbf{Q}\mathrm{e}^{\mathbf{Z}t}\mathrm{d}t =$$

$$= \mathrm{e}^{\mathbf{Z}^\mathsf{T}T} \int_0^T \mathrm{e}^{-\mathbf{Z}^\mathsf{T}(T-t)}\mathbf{Q}\mathrm{e}^{\mathbf{Z}t}\mathrm{d}t$$

We denote

$$\boldsymbol{\Psi}(t) = \mathrm{e}^{\mathbf{Z}t},$$

and formulate a system of equations

$$\begin{bmatrix} \dot{\boldsymbol{\Psi}}(t) \\ \dot{\mathbf{X}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{Z} & \mathbf{0} \\ \mathbf{Q} & -\mathbf{Z}^\mathsf{T} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}(t) \\ \mathbf{X}(t) \end{bmatrix}, \tag{15}$$

$$\begin{bmatrix} \boldsymbol{\Psi}(0) \\ \mathbf{X}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n\times n} \\ \mathbf{0}_{n\times n} \end{bmatrix}$$

then

$$\mathrm{e}^{\mathbf{Z}^\mathsf{T}T} \int_0^T \mathrm{e}^{-\mathbf{Z}^\mathsf{T}(T-t)}\mathbf{Q}\mathrm{e}^{\mathbf{Z}t}\mathrm{d}t =$$

$$= \boldsymbol{\Psi}(T)^\mathsf{T}\mathbf{X}(T)$$

One can see that if

$$\mathbf{G} = \begin{bmatrix} \mathbf{Z} & \mathbf{0} \\ \mathbf{Q} & -\mathbf{Z}^\mathsf{T} \end{bmatrix}$$

then

$$\mathrm{e}^{\mathbf{G}t} = \begin{bmatrix} \mathrm{e}^{\mathbf{Z}t} & 0 \\ \boldsymbol{\Xi}_1(t) & \boldsymbol{\Xi}_2(t) \end{bmatrix}$$

and

$$\boldsymbol{\Psi}(T)^\mathsf{T}\mathbf{X}(T) = \mathrm{e}^{\mathbf{Z}^\mathsf{T}T}\boldsymbol{\Xi}_1(T)$$

Finally denoting

$$\mathrm{e}^{\mathbf{Z}^\mathsf{T}T}\boldsymbol{\Xi}_1(T) = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_3 & \mathbf{P}_4 \end{bmatrix}$$

we get

$$\int_0^T x_n(t)^2\mathrm{d}t = \gamma^2\boldsymbol{\alpha}^\mathsf{T}\mathbf{P}_4\boldsymbol{\alpha}$$

and from (3) we have the formula on γ

$$\gamma = \pm\sqrt{\frac{E}{\boldsymbol{\alpha}^\mathsf{T}\mathbf{P}_4\boldsymbol{\alpha}}} \tag{16}$$

Numerical experiments and discussion

Below we present results of numerical computations for $n = 1, 2, 3, 4$. Figure 3 presents how control changes with the rise of n. In figures 4 - 7 the state variables for different n are presented. As one can easily see the controls for differen n are smooth and similar to one another. Also similar trends are visible also in state trajectories.

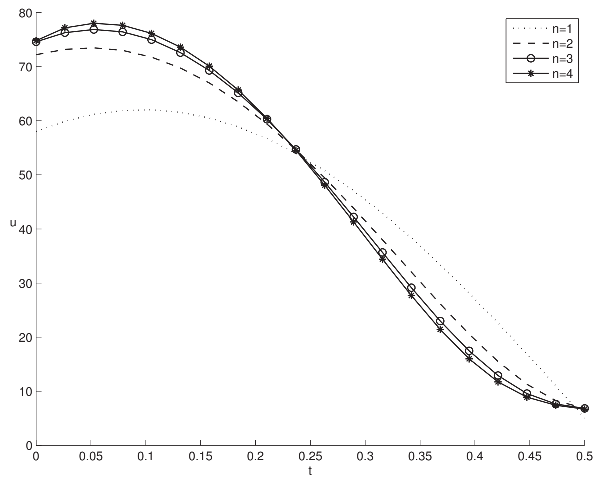

Fig. 3. Controls for different n

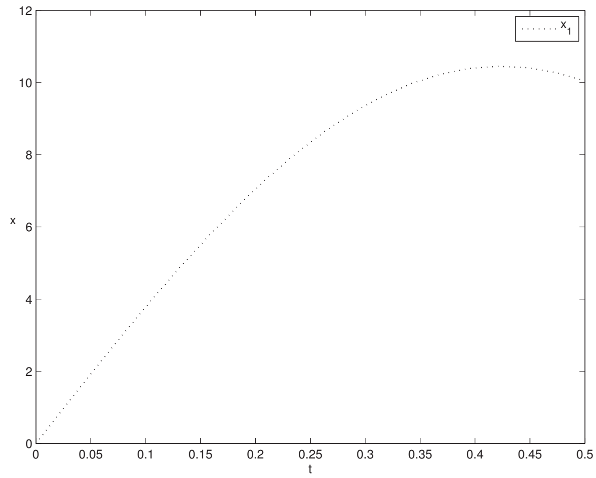

Fig. 4. Evolution of x for $n = 1$

Some discussion is however needed regarding the numerical computation. For $n = 1$ all computations can be performed analytically. Namely equation (12) becomes (see [20])

$$\mathrm{tg}(\omega(\rho)T) = \frac{\omega(\rho)}{Z_1}$$

where

$$\omega(\rho) = \sqrt{\left| Z_1^2 + \left(2\rho - \frac{1}{2}\right) Z_2 \right|},$$

$$Z_1 = \frac{2R_1 + R_H + 3R/2}{2C(R_1 + R/2)(R_H + R/2)},$$

$$Z_2 = \frac{1}{2C^2(R_1 + R/2)^2},$$

this equation has infinite number of solutions, however it can

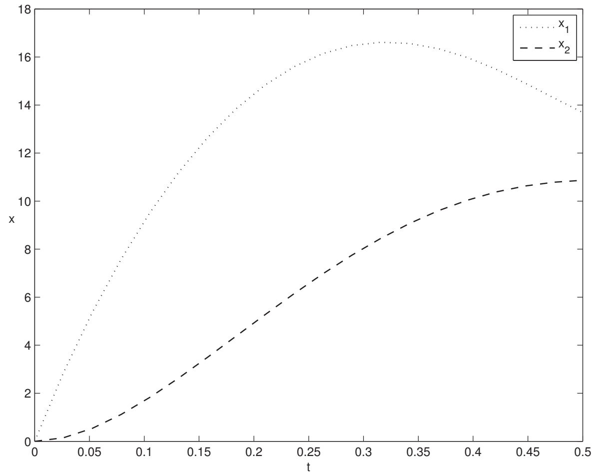

Fig. 5. Evolution of state variables for $n = 2$

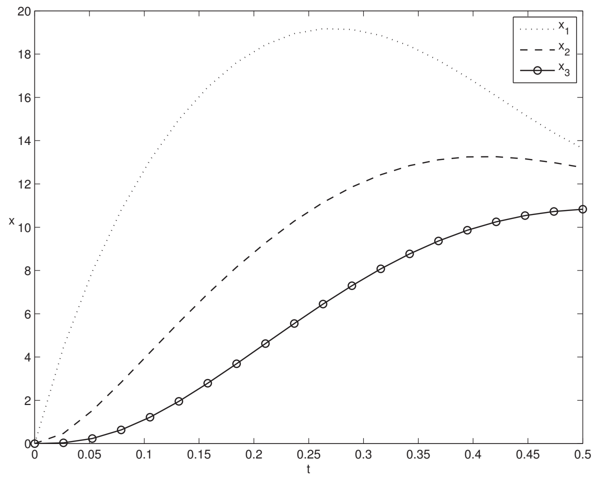

Fig. 6. Evolution of state variables for $n = 3$

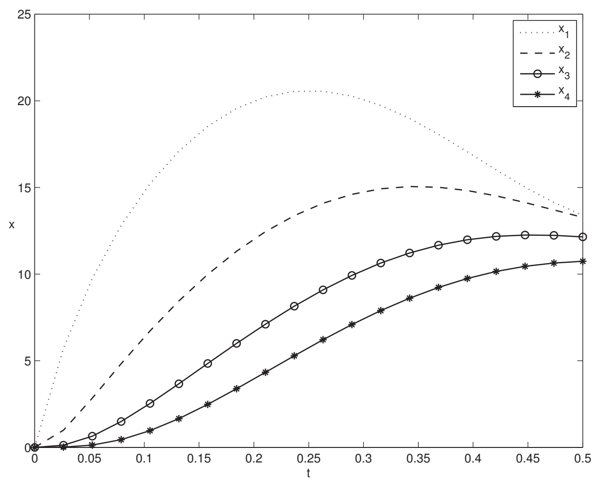

Fig. 7. Evolution of state variables for $n = 4$

be shown after tiresome computation, that

$$\psi_1^*(0) = \pm\frac{(R_H+R/2)\omega^*}{Z_2}\sqrt{\frac{2E\left(1+\left(\frac{\omega^*}{Z_1}\right)^2\right)}{R_H T\left(1-\frac{1}{Z_1T}+\left(\frac{\omega^*}{Z_1}\right)^2\right)}}$$

where ω^* is the smallest positive solution of

$$\mathrm{tg}(\omega T) = \frac{\omega}{Z_1}$$

and optimal control is given by the following formula:

$$u^*(t) = \frac{1}{2}\left[B\Phi_4^*(t) + \Phi_2^*(t)\right]\psi_1^*(0),$$
$$B = \frac{2R}{(2R_1+R)RC}$$
$$\Phi_2^*(t) = \frac{Z_2}{\omega^*}\sin\omega^* t$$
$$\Phi_4^*(t) = \cos\omega^* t - \frac{Z_1}{\omega^*}\sin\omega^* t$$

Optimal trajectory is given by

$$x_1^*(t) = \Phi_2^*(t)\psi_1^*(0)$$

The case of $n = 1$ is however the only one where analytical solution can be obtained completely without the use of computers. For higher orders some numerical computation is needed. For $n = 2, 3$ equation (12) can be solved directly however problem becomes increasingly badly scalled. Eigenvalues of matrix $\mathbf{Z}$ increase their absolute values by the multiplicity of n^2. And because they are evenly spread among positive and negative ones the matrix exponential $\mathrm{e}^{\mathbf{Z}}$ becomes ill conditioned. For $n = 4$ determinant becomes useless for computation and problem can be solved only by reduction of the smallest eigenvalue to zero. For $n > 5$ problem is so badly conditioned that the algorithm is unable to solve it. There can be certain attempts to improve the conditioning, however in authors' opinion direct optimisation methods could lead to better results.

Conclusions

The problem of optimal energy transfer was considered and a semi analytical solution was presented. Application of classical results of control theory leads to potentially effective formulas and is an alternative to non deterministic methods such as evolutionary algorithms [11].

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