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The structural analysis of singularly perturbed state models of electronic circuits using formula templates

Streszczenie. *W artykule jest przeanalizowane pochodzenie małych parametrów w syngularnie zaburzonych równaniach różniczkowych opisujących stany układu elektronicznego. Udowodniono, że pochodzenie omówionych parametrów może być dwóch rodzajów - albo wskutek obecności składników układu o małych wartościach, albo wskutek pewnych relacji liczbowych między wartościami tych składników. Proponowane jest podejście do budowy i analizy strukturalnej modeli matematycznych z zaburzeniami syngularnymi na podstawie budowy tzw. szablonów wzorów.* ***Analiza pochodzenia małych parametrów w syngularnie zaburzonych równaniach różniczkowych opisujących stany układu elektronicznego***

Abstract. *The paper deals with the origin of small parameters in the singularly perturbed state models of electronic circuits. As is shown in the paper, there are at least two origins of small parameters - small values of circuit components and certain relationships between the values of circuit components which need not be small in magnitude. The approach is proposed to building and structural analysis of singularly perturbed mathematical models based on the formula templates, as they are called.*

Słowa kluczowe: układy elektroniczne, symulacja, modele matematyczne, zaburzenia syngularne, szablony strukturalne.
Keywords: electronic circuits, simulation, mathematical models, singular perturbations, structural templates.

Introduction

Qualitative analysis of dynamical systems without finding the solutions themselves have a profound impact in revealing fundamental relations between the particular behavior of the mentioned systems and specific properties of mathematical models. As to electronic circuits, qualitative research makes it possible to investigate the dependence of the behavior of solutions on parameters of the circuit, what is of great importance in some cases, for example, when parameter variations may cause essential unwanted changes in the circuit behavior. The mentioned phenomena can be observed, say, in electronic circuits whose behavior is described by the set of singularly perturbed differential state equations.

This paper deals with the structural analysis of singularly perturbed state models of linear electronic circuits with lumped parameters. The basis for the analysis is the set of formula templates, as they are called, allowing us to establish direct relations between the structural properties of the circuit and matrix coefficients of the state model used in the analysis.

Structural properties of state equations of circuits with lumped parameters

The state equations of an electronic circuit with lumped parameters can be presented in the form of differential and algebraic equations as follows [1]:

(1) $M\frac{dx}{dt} = Ax + Bv + B'\frac{dv}{dt}$;

a) $y = Cx + Dv,$

where $x \in \boldsymbol{R}^n$ is the *n*-dimensional real vector of state variables, $v \in \boldsymbol{R}^m$ is the m-dimensional real vector of input signals of independent voltage (E-components) and current (J-components) sources, $y \in \boldsymbol{R}^k$ is the k-dimensional vector of voltages and currents at the circuit outputs, t is time, and M, B', A, B, C, and D are matrix coefficients. As is known [2], equations (1,a) are called the singularly perturbed differential equations if the determinant of matrix M tends to zero: $\det M \to 0$. Mathematical aspects of the analysis of singularly perturbed state differential equations have been investigated in detail [2, 3]. The paper deals with the analysis of circuit structural properties causing the availability of singularities in (1). The idea of templates discussed in the paper is readily apparent from the fact that the structure of matrix coefficients in (1) points to specific features of the structure of the circuit analyzed. With this in mind let us recall the steps of formation of state equations in form (1). The formation of (1) includes the following steps [1]:

1. The development of the structural graph Γ of the circuit.
2. The formation of the graph tree allowing the division of graph edges between two sets - the set of tree branches T and set of chords N.
3. The formation of topological equations corresponding to Kirchhoff's laws for currents and voltages:

a) $\Pi \cdot I = 0,$

(2) b) $P \cdot V = 0,$

where I is the vector of edge currents and V is the vector of voltages across edges, Π is the matrix of independent cutsets, and P is the matrix of independent loops in the Γ graph. The rows of the Π matrix correspond to the tree branches and the columns correspond to tree branches and chords, respectively. And the rows of the P matrix correspond to the chords and the columns correspond to the chords and tree branches, respectively.

Assume that each independent cutset includes no more than one tree branch, and each independent loop includes no more than one chord. The above cutsets and loops are called the principal cutsets and loops, or the principal system of coordinates.

We can represent vectors I and V in the form of two subvectors: the subvector of currents (voltages) of tree branches and chords: $I = (I_T,\ I_N)^T$, $V = (V_T,\ V_N)^T$, where subscripts *T* and *N* denote tree branches and chords, respectively, and superscript T denotes vector transposition. If the principal system of coordinates is chosen, matrix Π in (2a) is split in two submatrices – the identity submatrix of principal cutsets for tree branches E_T and the submatrix of principal cutsets for chords π_N: $\Pi = [E_{T,}\ \pi_N]$. And dually, matrix P in (2b) can be represented in the form of two submatrices – the submatrix of principal loops for tree branches ρ_T and the identity submatrix of principal loops for chords E_N : $P = [\rho_T,\ \mathrm{E}_N]$. Taking into account the above comments, we can rewrite (2) in the form:

$$\text{a) } [\mathrm{E}_T,\ \pi_N] \cdot \begin{bmatrix} \mathrm{I}_T \\ \mathrm{I}_T \end{bmatrix} = 0,$$

$$\text{(3) b)} [\rho_T,\ \mathrm{E}_N] \cdot \begin{bmatrix} \mathrm{V}_T \\ \mathrm{V}_N \end{bmatrix} = 0.$$

It is easy to verify that the following relationships are valid: $\pi_N = -\rho_T^T,\ \ \rho_T = -\pi_N^T$, where subscripts T and N denote tree branches and chords, respectively, and superscript T denotes matrix transposition. In our following discussion, we use matrix π_N, but it is evident that any relationship expressed in terms of the π_N matrix can be rewrite in terms of the ρ_T matrix.

4. The formation of component equations, which express relationships between currents and voltages in each circuit component. For the sake of simplicity, we restrict our consideration to the current-voltage relationships only, but it is possible to extend our classification to other kinds of circuit variables, for example, such as charge-current and so on.

Taking into account the fact that any multi-port circuit component is represented in graph $\Gamma$by certain subgraph consisting of two-terminal edges, we can reduce our consideration to circuits containing two-terminal circuit components. Each two-terminal circuit component can be assigned to one of two classes: y– or z–components. If the current (voltage) variable is subject to other circuit variables in the component equation, then this component is assigned to the class of y–components (z-components). According to this classification, the voltage controlled current source (VCCS) as well as the current controlled current source (CCCS) should be assigned to the class of y–components, because their component equations express current as a function of another circuit variable, namely: $I_{out} = sV_{in}$ and $I_{out} = nI_{in}$, respectively, where V_{in} and I_{in} are controlling voltages and currents of any circuit component considered as input variables for those controlled sources, I_{out} is the current through the given controlled source considered as the output variable, and s and n are controlling parameters. By the same token, the voltage controlled voltage source (VCVS) as well as the current controlled voltage source (CCVS) are considered as z-components, because component equations express voltage across those sources as a function of other circuit variables (voltage and current, respectively): $V_{out} = mV_{in}$ and $V_{out} = rI_{in}$, where m and r are controlling parameters. The independent current source J and the independent voltage source E are classified as the y- and z-components, respectively. The short-circuited component, whose resistance is zero $R = 0$, is the z-component, because its component equation is given by the equation $V_R = 0$, and the circuitry with zero conductance $G = 0$ is the y-component, because its component equation is given by the equation I_G=0. The resistor with non-zero resistance R≠0 can be assigned either to y– or z-component, and is called a dual component. Formally, capacitances and inductances might by classified as dual components, too, because their component equations can be represented either as currents as functions of voltages or voltages as functions of currents. But in order to represent the state model in the form of differential and algebraic equations (1), capacitances and inductances should be assigned to the classes of y– and z–components, respectively, because under this assumption, we use component equations in the differential forms: $I_C = \frac{dV_C}{dt}$ and $V_L = \frac{dI_L}{dt}$.

Thus the representation of the state model in the form of the set of differential and algebraic equations (1) can be provided by the inclusion of all the voltage sources and maximum possible number of capacitances into the set of tree branches T, and by the inclusion of all the current sources and maximum possible number of inductances to the set of chords N. The tree built using this rule is called the normal tree of the structural graph Γ.

If graph Γ contains a loop consisting of capacitances and maybe voltage sources, then one of the capacitances should be assigned to the set of chords, and if graph Γ contains a cutset consisting of inductances and maybe current sources, then one of the inductances should be assigned to the set of tree branches. The mentioned loops and cutsets are called the degenerate loops and cutsets, respectively. As is known [1], the availability of degenerated loops and cutsets does not bring into change in the general structure of model (1), since any capacitance voltage in the degenerate loop can be expressed as a sum of voltages of the rest components of the loop. In a similar fashion, any inductance current in the degenerate cutset can be expressed as a linear sum of currents of the rest components of the cutset.

5. The substitution of component equations into the set of topological equations (3) yields the state equations in the form (1), in which the state variable vector x consists generally of the subvector of voltages of the capacitance branches included into the normal tree V_{CT} and the subvector of currents of the inductive chords I_{LN}: $x = (V_{CT}, I_{LN})^T$. Vector v in (1) consists generally of the subvector of voltages of independent voltage sources V_E and currents of independent current sources I_J, that is, $v = (V_E,\ I_J)^T$,where the superscript T is the transposition symbol.

Following the above steps, we can form some templates for filling in matrix M in (1) with nonzero elements and use these templates to analyze structural properties of circuits, whose behavior is described by a singularly perturbed state equations.

Using templates for filling in matrix *M* with nonzero elements in the analysis of singularly perturbed state equations

Let us denote rows and columns of matrix coefficients by the same symbols as the corresponding circuit elements. For example, $\pi_{LS,Li}$ is the element of the π_N matrix located in the row denoted by symbol L_s and in the column denoted by symbol L_i. If $\pi_{LS,Li} \neq 0$, then graph Γ contains a degenerate cutset, which includes the inductive tree branch L_s and the inductive chord L_i. This situation takes place when the L_i chord is incident to the principal loop, defined by the L_s tree branch. Other notations presented below can be interpreted in a similar way.

We list below some typical templates for filling in matrix M in (1) with nonzero elements, which were derived using the sequence of steps described in the previous Section. The notation used in our relationships makes it possible to interpret circuit structural properties with reasonable facility. For example, the expression presented as template 4 is of the form

$M_{Li,Lj} = M_{Lj,Li} = \pi_{Ls,Li} \cdot \pi_{Ls,Lj} \cdot L_s$.

It means that the value of the L_s inductance is written in row L_i and column L_j of matrix M (and symmetrically, in row L_j and column L_i) with the sign defined by the product of the two unity elements of submatrix π_N, namely, $\pi_{Ls,Li}$ and $\pi_{Ls,Lj}$.

Examples of templates for filling in matrix M with nonzero elements:

1. *Structural conditions*: The capacitive edge C_i belongs to the normal tree $C_i \in T$.
   Template: $M_{Ci,Ci} = C_i$.
2. *Structural conditions*: The inductive edge L_j belongs to the set of chords $L_j \in N$.
   Template: $M_{Lj,Lj} = L_j$.
3. *Structural conditions*: The inductivities $L_j \in N$ and $L_s \in N$ are related by mutual inductance L_{js}.
   Template: $M_{Lj,Ls} = M_{Ls,Lj} = -L_{js}$.
4. *Structural conditions:* The inductive tree branch $L_s \in T$ is incident to the degenerate cutset that includes also inductive chords $L_i, L_j \in N$.
   Template:
   $$M_{Li,Lj} = M_{Lj,Li} = \pi_{Ls,Li} \cdot \pi_{Ls,Lj} \cdot L_s.$$
5. *Structural conditions:* The capacitive chord $C_t \in N$ is incident to the degenerate loop that includes also capacitive tree branches $C_j, C_s \in T$.
   Template: $M_{Cj,Cs} = M_{Cs,Cj} = \pi_{Cj,Ct} \cdot \pi_{Cs,Ct} \cdot C_t$.
6. *Structural conditions:* The degenerate cutset includes the VCCS, $Q_t \in N$, the inductive chord $L_j \in N$, and the inductive tree branch $L_k \in T$.
   Templates: 6.1. $V_{in} = V_{Ci}$, where V_{Ci} is the controlling voltage across the capacitive tree branch $C_i \in T$:
   $$M_{Lj,Ci} = \pi_{Lk,Lj} \cdot \pi_{Lk,Qt} \cdot s \cdot L_k.$$
   6.2. $V_{in} = V_{Ci}$, where V_{Ci} is the controlling voltage across the capacitive chord $C_i \in N$, which is incident to the degenerate loop containing the capacitive tree branch $C_t \in T$:
   $$M_{Lj,Ct} = -\pi_{Lk,Lj} \cdot \pi_{Lk,Qt} \cdot \pi_{Ct,Ci} \cdot s \cdot L_k.$$
7. *Structural conditions:* The degenerate cutset includes the CCCS, $Q_t \in N$, the inductive tree branch $L_k \in T$, and the inductive chord $L_j \in N$.
   Templates: 7.1. $I_{in} = I_{Li}$, where I_{Li} is the controlling current through the inductive chord, $L_i \in N$: $M_{Lj,Li} = \pi_{Lk,Lj} \cdot \pi_{Lk,Qt} \cdot n \cdot L_k$.
   7.2. $I_{in} = I_{Li}$, where I_{Li} is the controlling current through the inductive tree branch $L_i \in T$, which is incident to the degenerate cutset containing also the inductive chord $L_t \in N$:
   $$M_{Lj,Lt} = -\pi_{Lk,Lj} \cdot \pi_{Lk,Qt} \cdot \pi_{Li,Lt} \cdot n \cdot L_k.$$
8. *Structural conditions:* The degenerate loop includes the VCVS, $U_t \in T$, the capacitive tree branch, $C_j \in T$, and the capacitive chord $C_t \in N$.
   Templates: 8.1. $V_{in} = V_{Cs}$, where V_{Cs} is the controlling voltage across the capacitive tree branch $C_s \in T$: $M_{Cj,Cs} = \pi_{Cj,Ct} \cdot \pi_{Ut,Ct} \cdot m \cdot C_t$.
   8.2. $V_{in} = V_{Cs}$, where V_{Cs} is the controlling voltage across the capacitive chord $C_s \in N$, which is incident to the degenerate loop containing also the capacitive tree branch $C_i \in T$:
   $$M_{Cj,Ci} = -\pi_{Cj,Ct} \cdot \pi_{Ut,Ct} \cdot \pi_{Ci,Cs} \cdot m \cdot C_t.$$
9. *Structural conditions:* The degenerate loop includes the CCVS, $U_t \in T$, the capacitive tree branch $C_j \in T$, and the capacitive chord $C_t \in N$.
   Templates: 9.1. $I_{in} = I_{Ls}$, where I_{Ls} is the controlling current through the inductive chord, $L_s \in N$:
   $$M_{Cj,Ls} = \pi_{Cj,Ct} \cdot \pi_{Ut,Ct} \cdot r \cdot C_t.$$
   9.2. $I_{in} = I_{Li}$, where I_{Li} is the controlling current through the inductive tree branch, $L_i \in T$, which is incident to the degenerate cutset containing also the inductive chord $L_s \in N$:
   $$M_{Cj,Ls} = -\pi_{Cj,Ct} \cdot \pi_{Ut,Ct} \cdot \pi_{Li,Ls} \cdot r \cdot C_t.$$

Comments: (1) s, r, m, and n are controlling parameters of controlled sources; (2) versions 6, 7, 8, and 9 include two possible situations - the controlling edge of the controlled source is assigned to the set of branches of the normal tree T or to the set of chords N.

Of course, templates given above do not exhaust all the possible circuit configurations, but provide insight into the main idea of the method proposed. They exemplify the fact that the use of templates, which are simple to be programmed, may save as a convenient tool for the analysis of structural features of singularly perturbed state models.

It is evident that in the simplest case, when the structural graph Γ of an electronic circuit is free of structural degenerations, matrix M maybe singular if small values of reactive circuit components – capacitances and/or inductances – are present in the matrix diagonal elements. It is a rather trivial case, which does not require additional comments.

In more sophisticated cases, namely, when the circuit graph contains structural degenerations, matrix M may be singular because of certain relationships between the values of circuit components, which are not necessarily small in magnitudes. The structural properties of the circuit, whose state model is singular, can be easily stated using templates for filling in the M matrix with nonzero elements given above. This thesis is illustrated in the next simple example.

Example 1. Let us consider the circuit consisting of the independent voltage source E, resistive components R_1 and R_2, capacitances C_1, C_2, and C_3, and the voltage controlled voltage source (VCVS) U (Fig.1), whose component equation is as follows: $V_U = mV_{C1}$, where V_{C1} is the controlling voltage across the C_1 capacitance and m is the controlling parameter.

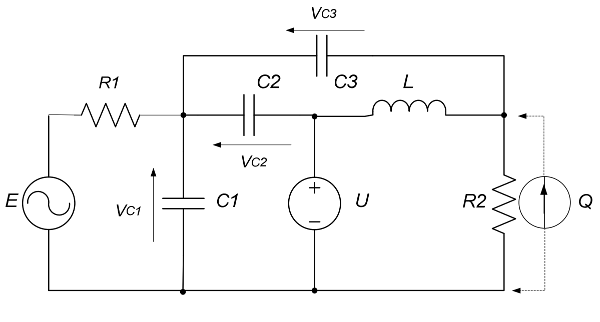

Fig.1. A simple circuit containing a degenerate loop *C1-C2-U*

Figure 2 presents the structural graph of the circuit, in which normal tree branches are pictured by bolt edges, chords by thin edges, and principal cutset by dotted lines.

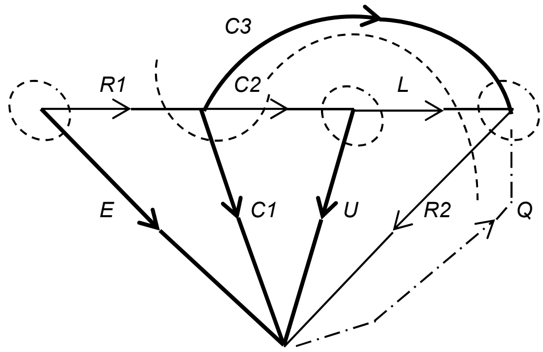

Fig.2. The structural graph Γ of the circuit with the selected branches of the normal tree T (bolt lines), chords N (thin lines),principal cutsets (dotted lines), and dot-and-dash chord *Q*

The matrix of principal cutsets Π is of the form:

$\Pi = [E_T,\ \pi_N]$ =

	E	C_1	U	C_3	R_1	C_2	L	R_2
E	1				1			
C_1		1			−1	1	−1	1
= U			1			−1	1	
C_3				1			1	−1

According to the selected normal tree, the state vector *x* consists of the two voltages across the tree capacitances C_1 and C_3 and the current through the inductive chord L: $x = (V_{C1}, V_{C3},\ I_L)^T$. Thus matrix M in (1) consists of three rows and columns, which we denote by symbols C_1, C_3, and L. To form the M matrix, we can use the templates given above. Using templates 1 and 2, we write elements C_1, C_3, and L in the principal diagonal.

In view of the presence of a nonzero element in row C_1 and column C_2 in matrix Π, namely, $\pi_{C1,C2} = 1$, we also use template 4. And taking into account that in our example C_j, $C_s \equiv C_1$, and $C_t \equiv C_2$, we write the next element in M: $M_{C1,C1} = \pi_{C1,C2} \cdot \pi_{C1,C2} \cdot C_2 = 1{\cdot}1{\cdot}C_2 = C_2$. Finally, because of the presence of the degenerate loop consisting of components U, C_1, and C_2, and taking into account that $V_{in} = V_{Cs} = V_{C1}$, and $C_j = C_1$ and $C_t = C_2$, we use template 8.1 to write the element $M_{C1,C1}$ in M: $M_{C1,C1} = \pi_{C1,C2} \cdot \pi_{U,C2} \cdot m \cdot C_2 = 1{\cdot}(-1){\cdot}m{\cdot}C_2 = -m{\cdot}C_2$. Thus matrix M takes the form of the diagonal matrix $\mathrm{diag}[C_1 + C_2\ (1-m),\ C_3,\ L]$. It is easy to verify that following the above procedure of the formation of the state model, we can obtain equations (1,a) in the form:

$$(4)\qquad \begin{bmatrix} C_1 + C_2(1-m) & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & L \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_{C1} \\ V_{C3} \\ I_L \end{bmatrix} = \begin{bmatrix} -(G_1+G_2) & G_2 & -1 \\ G_2 & -G_2 & -1 \\ m-1 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C3} \\ I_L \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \\ 0 \end{bmatrix} V_E .$$

It is evident that the determinant of matrix M approaches zero if one or more diagonal elements tend to zero. In the simplest case, it is possible, for example, when C_1, $C_2 \rightarrow 0$, and/or $C_3 \rightarrow 0$, or/and $L \rightarrow 0$. In more complicated case, $\det M \rightarrow 0$ if $m \rightarrow (C_1 + C_2) / C_2$ for any nonzero values of capacitances C_1 and C_2 and controlling parameter m. If we take that $m = (C_1 + C_2)/C_2$, then the first row will consist of zeros. In this case, the first equation in (4) becomes an algebraic equation of the form

$$0 = -(G_1 + G_2) \cdot V_{C1} + G_2 \cdot V_{C3} - I_L + G_1 \cdot V_E,$$

and any state variable may be expressed as a linear function of other state variables. For example, we can write

$$V_{C1} = (G_1 + G_2)^{-1} \cdot (G_2 \cdot V_{C3} - I_L + G_1 \cdot V_E).$$

Substituting this into the second and third equations, we can exclude variable V_{C1} from equations (4). Therein lies the simplest method of the reduction of singularly perturbed differential equations which we discuss below in more detail.

Now, let us change slightly our problem. Assume that the VCVS is controlled by the voltage across the C_3 capacitance: $V_U = mV_{C3}$. To write parameter m in matrix M, we can use again template 8.1, but now we take into account that $C_j = C_1$, $C_t = C_2$, and $C_s = C_3$. Consequently, we can form the following template:

$$M_{Cj,Cs} = M_{C1,C3} = \pi_{Cj,Ct} \cdot \pi_{Ut,Ct} \cdot m \cdot C_t = \pi_{C1,C2} \cdot \pi_{U,C2} \cdot m \cdot C_t = 1{\cdot}(-1){\cdot}m{\cdot}C_2 = -mC_2.$$

In this case, matrix *M* takes the form:

$$(5)\qquad M = \begin{bmatrix} C_1 + C_2 & -mC_2 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & L \end{bmatrix}$$

It is evident that matrix M will not be singular for any nonzero positive and reasonable elements.

Of course, they are very simple examples. In more sophisticated cases, the singularly perturbed state equations may be caused by the presence of certain off-diagonal elements in matrix M and linear relationships of some rows. To illustrate this statement, let us change our example once more including CCCS in parallel to R_2, as is shown in Fig. 1 by the dash line. Let the component equation of the CCCS be as follows: $I_Q = nI_{C2}$, where I_{C2} is the controlling current through the capacitance C_2 and n is the controlling parameter.

The structural graph Γ is augmented by chord Q, which is represented by the dot-and-dash line in Fig.2. Following the above technique, we can obtain the templates for filling in matrix M:

$$M_{C1,C1} = C_1 + \pi_{C1,C2} \cdot \pi_{C1,C2} \cdot C_2 + \pi_{C1,Q} \cdot \pi_{C1,C2} \cdot n \cdot C_2 =$$
$$= C_1 + 1 \cdot 1 \cdot C_2 + (-1) \cdot 1 \cdot n \cdot C_2 =$$
$$C_1 + C_2(1-n);$$
$$M_{C1,C3} = \pi_{C1,C2} \cdot \pi_{U,C2} \cdot m \cdot C_2 + \pi_{C1,Q} \cdot \pi_{U,C2} \cdot m \cdot n \cdot C_2 =$$
$$= 1 \cdot (-1) \cdot m \cdot C_2 + (-1) \cdot (-1) \cdot m \cdot n \cdot C_2 =$$
$$m \cdot C_2 \cdot (n-1);$$
$$M_{C3,C1} = \pi_{C3,Q} \cdot \pi_{C1,C2} \cdot n \cdot C_2 = 1 \cdot 1 \cdot n \cdot C_2 =$$
$$n \cdot C_2;$$
$$M_{C3,C3} = C_3 + \pi_{C3,Q} \cdot \pi_{U,C2} \cdot m \cdot n \cdot C_2 =$$
$$C_3 + 1 \cdot (-1) \cdot m \cdot n \cdot C_2 =$$
$$C_3 - m \cdot n \cdot C_2;$$
$$M_{L,L} \quad = L.$$

Hence, matrix M will look like:

$$(6)\quad M = \begin{bmatrix} C_1 + C_2(1-n) & mC_2(n-1) & 0 \\ nC_2 & C_3 - mnC_2 & 0 \\ 0 & 0 & L \end{bmatrix}.$$

It is easy to verify that M becomes singular if $n = C_3(C_1 + C_2)/C_2(mC_1 + C_3)$. Indeed, substituting this into (6), we obtain matrix M as follows:

$$(7)\quad M = \begin{bmatrix} \frac{mC_1(C_1+C_2)}{mC_1+C_3} & \frac{mC_1(C_3-mC_2)}{mC_1+C_3} & 0 \\ \frac{C_3(C_1+C_2)}{mC_1+C_3} & \frac{C_3(C_3-mC_2)}{mC_1+C_3} & 0 \\ 0 & 0 & L \end{bmatrix}.$$

Next, multiplying the second row by $mC_1 \,/\, C_3$ and adding it to the first row, we obtain matrix M with zero row. Just as in the first case, it means that one of the state variables can be expressed as a linear function of other state variables, and the set of state equations can be reduced.

What the first and the third versions of our example have in common is that the matrix M singularity is caused by certain relationships between the values of circuit components, but not by small component values. It becomes possible due to the existence of structural degeneration in the circuit. To clear up the conditions under which the mentioned singularities occur, we can use the templates for filling in matrix M with nonzero elements, whose examples are present above. In addition, we ensured that it is possible to reduce the set of differential state equations by way of zeroing small parameters in matrix M. The question arises of whether the mentioned reduction is mathematically correct and which specific features exhibit singularly perturbed state models that should invite our attention? The theory of singularly perturbed sets of differential equations provides an answer to this question.

On mathematical features of singularly perturbed state equations

Let us bring briefly the issue of singularly perturbed state models and dwell on the some specific features of the mentioned models which differentiate them from “ordinary” models.

The singularly perturbed set of ordinary differential equations (ODE) has been the subject of some mathematical studies for a long time. We would like to mention the results concerning this subject obtained by A. Tikhonov (see, for example, [3]). He studied the Cauchy problem of the form:

$$(8)\quad \begin{cases} a)\ \mu\dot{x} = f(x,y), & x(0) = x^0, \ \ x \in \mathbf{R}^n, \\ b)\ \dot{y} = g(x,y), & y(0) = y^0, \ \ y \in \mathbf{R}^m, \end{cases}$$

where x and y are the n-dimensional and m-dimensional subvectors of state variables, respectively, determined in real spaces, and μ is the diagonal matrix of parameters, whose magnitudes are small enough (say, with respect to unity). The initial conditions for state variables x and y at the initial time point $t_0 = 0$ are given by vectors x^0 and y^0. According to Tikhonov’s theory, equations (8,a) and (8,b) exhibit processes with relatively fast and relatively slow rates, respectively, and therefore variables x and y are called fast and slow variables. The components of subvector x vary mainly on the relatively narrow boundary layer in the vicinity of the initial time $t \in [0\ ,\ \tau_b]$ and the components of subvector y are changed mainly beyond this layer. If matrix μ contains small parameters with different magnitudes, then the boundary layer is split into a series of boundary sublayers, each being characterized by its own extent.

For reasonably small values of parameters μ the set of equations (8) possesses the stiffness property, which complicates numerical solution of equations [4]. One of the ways to simplify the problem is to set small parameters μ to zero. Following such a simplification, model (8) is reduced to the form of the set of algebraic and differential equations:

$$(9)\quad \begin{cases} a)\ 0 = f(x,y), \ \ x \in \mathbf{R}^n, \\ b)\ \dot{y} = g(x,y), \ \ y(0) = y^0, \ \ y \in \mathbf{R}^m, \end{cases}$$

which can be solved using simpler in a certain sense numerical methods. We discussed above structural conditions under which such a reduction of the state model is possible. Comparing the perturbed and reduced state models, some specific mathematical properties should be considered.

First, reducing equations (8) to the form of equations (9), we ignore initial conditions $x(0)$, therefore the solution of (9) within the boundary layer may be different essentially from the solution of equations (8). Hence, such a model reduction is not acceptable in some cases when we need to analyze the circuit within the boundary layer. For example, very-large-scale integrated circuits (VLSIs) provide a good example of those complex engineering objects, whose small model parameters exhibit various second-order effects as they are called [5]. Zeroing small parameters, we exclude those phenomena from consideration and lose information on some fast-acting processes which take place in the actual object.

Second, as was shown in the above discussion, small coefficients on some derivatives of state variables may be caused not only by small values of some parameters, but also because of the particular relationships between component values of the object analyzed. Typically, the mentioned relationships point to some kind of functional relationships between the parts of a complex object, which may require closer examination. For example, lateral transistor effects between the components which are mounted close together on the substrate of a VLSI offer examples of this kind relationships.

And third, reducing model (8) we should be assure that the solution of the reduced set of equations approaches the solution of the solution of the perturbed set of equations at least beyond the boundary layer. As is known [3], Tikhonov's theory establishes certain sufficient conditions, under which the mentioned solutions approach asymptotically each other. Those conditions can be interpreted by the example of a simple case.

Assume that we deal with the set of singularly perturbed differential equations (8) consisting of two variables x and y, that is, $x \in \mathbf{R}^1$ and $y \in \mathbf{R}^n$. Zeroing parameter μ in (8,a) and resolving the obtained algebraic equation $f(x,y)=0$ with respect to variable x, we can represent x as a function of y: $x=\varphi(y)$. Substituting this into (8, *b*), then solving the obtained differential equation $\dot{y}=g(\varphi(y),y)$, $y(0)=y^0$ with respect to y, and substituting this to equation $x=\varphi(y)$, we can obtain the solution of the reduced set of equations (9). Let this solution be $(\bar{x},\bar{y})$. The question arises, which root of the equation $f(x,y)=0$ should be selected in the case when the mentioned equation is nonlinear and possesses a series of roots?

According to the mentioned theory, the solution of the set of equations $f(x,y)=0$ and $\dot{y}=g(\varphi(y),y)$, $y(0)=y^0$ approaches the solution of problem (8) within a certain attraction closed domain $(x,\ y)\in D$ if the root $x=\varphi(y)$ of equation $f(x,y)=0$ is stable, as it is called, and the initial point $(x^0,\ y^0)$ belongs to the domain D of the stable root $x=\varphi(y)$. If the mentioned conditions are met, then the solution of the set of perturbed equations (8) $x(t,\mu)$ and $y(t,\mu)$ approach asymptotically the solution of the reduced set of equations (9) $\bar{x}(t)$ and $\bar{y}(t)$ within certain time regions:

$$x(t,\mu)\to\bar{x}(t),\ \ \tau_b\le t\le T,$$

$$y(t,\mu)\to\bar{y}(t),\ \ 0\le t\le T, \tag{10}$$

where $[0,\tau_b]$ is the boundary layer.

However, if the above conditions are not met, the solution of the reduced set of equations (9) and the perturbed set of equations (8) may be far from each other as much as you like. The following simple example illustrates this fact.

Example 2. Let us consider the singularly perturbed equation:

$$\mu\frac{dx}{dt}=x(t^2-x+1),\ \ x(t_0)=x_0, \tag{11}$$

where $\mu>0$ is a small positive parameter.

The reduced equation obtained by setting $\mu=0$ is as follows: $0=x(t^2-x+1)$. This equation possesses two roots: (1) $x=0$ and (2) $x=t^2+1$. Intuition suggests that the two roots provide different results of approximation. Which root should be selected? According to Tikhonov's theory [3], the first root is unstable because $\frac{\partial[x(t^2-x+1)]}{\partial x}=t^2+1>0$ for x=0, and the second root is stable because $\frac{\partial[x(t^2-x+1)]}{\partial x}=-t^2-1<0$ for $x=t^2+1$.

Hence, if the initial point $(t_0,\ x_0)$ lies in the upper half-plane (that is, $x>0$), then the curve of the solution of (11) approaches smoothly the $x=t^2+1$ curve as is shown in Fig.3. And if the initial point $(t_0,\ x_0)$ lies in the lower half-plane, then the curve of the solution of the perturbed equation (11) and the curve of the solution of the reduced equation are moving farther apart.

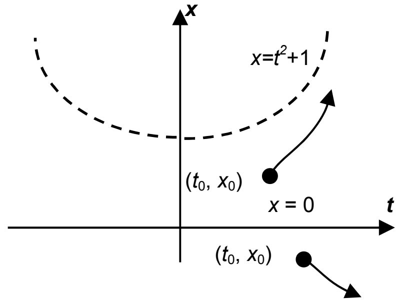

Fig.3. The curves of the solutions of the perturbed equation (continuous lines) and the reduced equation (dotted line)

It is evident that the mentioned problem may be the subject of special studies, whose aim is the development of special-purpose numerical methods of solution of singularly perturbed sets of state equations [4, 6].

And the final subject which we would like to touch on: is it possible to expand the ideas of structural analysis discussed in the paper on other classes of objects different from the class of electronic circuits? To confirm this suggestion, we would like to mention a well-known principle of physical analogies. As is known, there are certain correspondences between variables in different physical areas. By way of illustration, Table 1 represents analogies between the electrical and mechanical variables.

Table 1. Analogies between the electrical and mechanical variables.

Electrical variable	Mechanical variable
Inductance, *L*	Mass, *m*
Electrical charge, *q*	Linear displacement, *x*
Time, *t*	Time, *t*
Current, *i*	Linear velocity, *v*
Electromotive force, *e*	Force, f_M
Electrical resistance, *r*	Mechanical resistance, r_M
Electrical capacitance, *C*	Pliability, C_M

The mentioned analogies like those given in Table 1, suggest that the development of the above approach in other engineering areas is valid.

Conclusion

Singularly perturbed state equations form a particular class of mathematical models, whose investigation calls for the use of special-purpose numerical methods as well as qualitative methods allowing the designer establish specific structural properties of objects analyzed. It is customary to associate the phenomena of singularity with the existence of small parameters in the object model, whose zeroing does not involve substantial errors in numerical analysis.

Nevertheless, even simple examples discussed in the paper show that singularly perturbed state models are characterized by a series of specific properties, which should not be ignored in practical design of complex engineering objects. First, reduction of the state model by way of zeroing small parameters may lead to the model, whose behavior is essentially different form the behavior of the perturbed model. And second, except trivial cases when small component valuesare present in the circuit, model singularity may be caused by certain numerical relationships between the values of circuit components under proper structural conditions, namely, if the structural graph of the circuit contains structural degenerate cutsets and/or loops.The method of formula templates proposed in the paper allowing the designer to simplify the detection of the above structural conditions.

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