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An Application of ΣΔ Undersampling to an Estimation of Linear System's Impulse Response

Streszczenie**. Zaprezentowano metodę pomiaru odpowiedzi impulsowej układu liniowego wykorzystującą podpróbkowanie ΣΔ jego sygnału wyjściowego przy prostokątnym sygnale wejściowym. Przedstawiono algorytm przetwarzania sygnałów, umożliwiający redukcję błędów pomiarowych. Przeanalizowano zależność błędów od krotności podpróbkowania, liczby próbek oraz błędów częstotliwości na przykładzie 2 typowych układów liniowych.* ***Metoda pomiaru odpowiedzi impulsowej układu liniowego wykorzystującą podpróbkowanie ΣΔ

***Abstract**. A method of a measurement of a linear system's impulse response is presented. For a rectangular input signal ΣΔ undersampling of the system's output signal enables calculations of its impulse response. An algorithm of output signal's digital processing enabling a reduction of errors is described. A dependence of measurement's errors on the number of samples per period and relation between sampling and signal's frequencies was analyzed. Exemplary calculations were performed for two typical linear systems.*

Keywords: undersampling, a linear system, an impulse response, frequency fluctuations
Słowa kluczowe: podpróbkowanie, odpowiedź impulsowa, ΣΔ

An introduction

An impulse response is the basic characteristics of every linear system. Its direct measurement is possible only in an approximative way, because generation of Dirac's impulse is impossible. It might be achieved as a derivative of system's step response, however a construction of an ideal differentiating circuit is impossible too. Therefore a method, applying ΣΔ undersampling [4-6] is proposed. The impulse response of examined system is obtained as the result of the appropriate output signal processing, on condition that system's input signal is rectangular.

An idea of the method

When a rectangular wave of a frequency f_0 is applied as the input signal of the linear system, its output signal $x_0(t)$ is periodic and equal to the step response, on condition, that f_0 is low enough. Therefore it can be expressed in a form of Fourier's series

(1) $$x_0(t) = A_0 + \sum_{m=1}^{\infty} A_m \cdot \cos\left(2 \cdot \pi \cdot m \cdot f_0 \cdot t + \varphi_m\right).$$

A_0 denotes the mean value of the signal, A_m and φ_m - the amplitude and the phase of the mth harmonic component, respectively.

To obtain the impulse response $h_0(t)$ of the linear system, $x_0(t)$ should be differentiate.

(2) $$h_0(t) = \frac{dx_0(t)}{dt} = -\sum_{m=1}^{\infty} 2m\pi \cdot f_0 \cdot A_m \cdot \sin\left(2\pi m \cdot f_0 \cdot t + \varphi_m\right)$$

To apply the digital signal processing the output signal ought to be sampled. When the number of samples per period equals N, the k-th sample (k = 0 ... N-1) of $h_0(t)$ can be expressed as

(3) $$h_0(k) = -\sum_{m=1}^{\infty} 2 \cdot m \cdot \pi \cdot f_0 \cdot A_m \cdot \sin\left(\frac{2 \cdot \pi \cdot m \cdot k}{N} + \varphi_m\right).$$

Because FFT algorithm is used in further considerations, the number of samples must be a power of 2. To apply the ΣΔ undersampling, the time T_{d0} of integration of the output signal $x_0(t)$ must satisfy the following condition [1-6]

(4) $$T_{d0} = f_0^{-1} \cdot \left(M + N^{-1}\right),$$

where M is the undersampling factor.

A difference $y_0(t)$ of two successive integrals can be expressed as

(5)
$$y_0(t) = \int_t^{t+T_{d0}} x_0(t) \cdot dt - \int_{t-T_{d0}}^{t} x_0(t) \cdot dt =$$
$$= -\frac{2}{N \cdot f_0} \cdot \sum_{m=1}^{\infty} A_m \cdot \frac{\sin^2 \frac{m \cdot \pi}{N}}{\frac{m \cdot \pi}{N}} \cdot \sin\left(2m \cdot \pi \cdot f_0 \cdot t + \varphi_m\right)$$

Comparing (2) and (5) one can notice, that signals $h_0(t)$ and $y_0(t)$ are similar. Only the amplitudes of particular harmonic components of both signals differ one from another. Therefore the signal $h_0(t)$ can be obtained by means of filtering $y_0(t)$ by the filter, which transfer function $H_0(f)$ satisfies following conditions:

(6) $$X_0(m \cdot f_0) = H(m \cdot f_0) \cdot Y_0(m \cdot f_0),$$

(7) $$H(m \cdot f_0) = \frac{N^2 \cdot f_0^2}{\sin^2\left(\frac{m \cdot \pi}{N}\right)} \cdot \left(\frac{m \cdot \pi}{N}\right)^2.$$

The filter realizing conditions (6) and (7) can be designed as the finite response digital filter, which transfer function is given by following formulas:

(8) $$H(z) = \sum_{n=1}^{N-1} h(n) \cdot z^{-n},$$

(9) $$h(n) = (-1)^n \frac{\pi^2 N f_0^2}{4} + \frac{2 f_0^2}{N} \cdot \sum_{m=1}^{N/2-1} \frac{\left(\frac{m\pi}{N}\right)^2 \cos\frac{2mn\pi}{N}}{\sin^2\left(\frac{m \cdot \pi}{N}\right)}.$$

This filter also works as an anti-aliasing filter, because its attenuation at the frequency f_0 /2N exceeds 3 dB.

The method does not generate any errors itself on the condition, that frequencies of the signal and the sampling generator are exact and the synchronization between both generators is ideal. In an opposite case phases of measured signals change, therefore errors appear [6,7].

Relative stabilities of the signal's frequency and the sampling period are denoted as δf and δT_d respectively, so the real frequencies of the signal and the sampling period can be expressed as [7]:

(10) $$f = f_0 \cdot (1 + \delta f),$$

(11) $$T_d = T_{d0} \cdot (1 + \delta T_d).$$

The real values of samples of measured signal y(k) can be written as

(12) $$y(k) = -\frac{2}{N \cdot f_0} \cdot \sum_{m=1}^{\infty} A_m \cdot \frac{\sin^2 \frac{m \cdot \pi}{N}}{\frac{m \cdot \pi}{N}} \times$$
$$\times \sin\left[\frac{2 \cdot \pi \cdot m \cdot k(M \cdot N + 1)}{N} \cdot (1 + \delta\varphi) + \varphi_m\right]$$

(13) $$\delta\varphi = \delta f + \delta T_d + \delta f \cdot \delta T_d.$$

To avoid accidental errors of the measurement the following algorithm is applied [7]. Only single period of the signal $y_1(k)$ is measured and its FFT denoted Y(p,n) is calculated.

(14) $$Y(p,n) = \sum_{k=0}^{N-1} y(k) \cdot \exp\left(-j \cdot \frac{2 \cdot \pi \cdot k \cdot n}{N}\right)$$

This measurement is repeated P times. A mean value of y(k) is obtained as the geometric mean of all FFTs, afterwards IFFT is calculated, according to (15) and (16).

(15) $$\overline{Y}(n) = \left[\prod_{p=1}^{P} Y(p,n)\right]^{\frac{1}{P}}$$

(16) $$\overline{y}(k) = \frac{1}{N} \cdot \sum_{n=0}^{N-1} \overline{Y}(n) \cdot \exp\left(j \cdot \frac{2 \cdot \pi \cdot k \cdot n}{N}\right)$$

In the end, the mean signal is filtered by the filter H(k).

(17) $$\overline{x}(k) = \sum_{i=0}^{N-1} \overline{y}(i) \cdot h(k-i)$$

To estimate errors of the described method simulations repeated P = 100000 times were performed. The calculations were made separately for every harmonic compound of the signal. An error δ_1, defined as

(18) $$\delta_1 = \frac{\sum_{p=1}^{P} \sum_{k=0}^{N-1} \left\|\overline{x}(k) + 2m\pi f_0 A_m \cdot \sin\left(\frac{2m\pi k}{N} + \varphi_m\right)\right\|}{2 \cdot \pi \cdot m \cdot f_0 \cdot A_m \cdot P \cdot N}$$

was chosen as the criterion of an accuracy of the method. A probability density p(f) of frequency errors, given as [7]

(19) $$p(f) = \frac{1}{f_0 \cdot \delta f} \cdot rect\left(\frac{1}{f_0 \cdot \delta f} - f_0\right),$$

was taken into considerations, because in this case the largest values of errors were obtained. It was also assumed, that the time of the measurement is short enough, so the frequencies of both generators are constant. Results of calculations are shown on Figures 1 - 3.

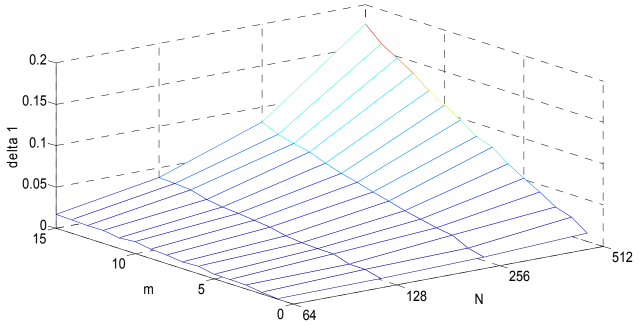

Fig. 1. A dependence of the error δ_1 on m and N for M=50, $\delta_f = \delta_{Td} = 10^{-6}$

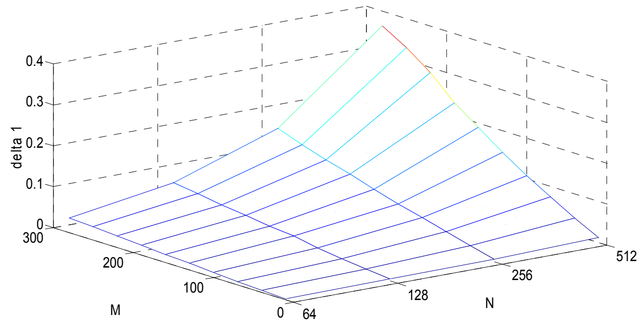

Fig. 2. A dependence of the error δ_1 on N and M for m=5, $\delta_f = \delta_{Td} = 10^{-6}$

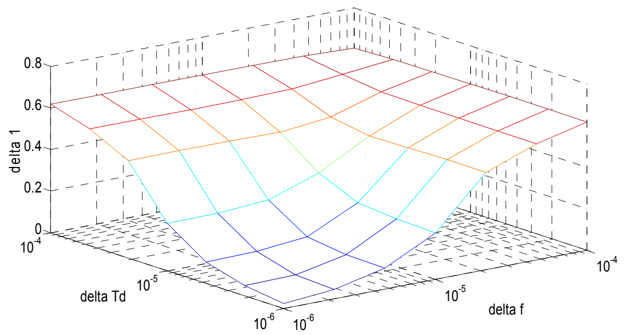

Fig. 3. The error δ_1 as the function of δ_f and δ_{Td} (m=5, M=50, N=256)

The results of the calculations prove, that the accuracy of the method is getting worse for greater values of N, M and the errors of both frequencies. The main reason of this effect is a summation of phase errors, when the time of the measurement gets longer. One should also notice, that relative errors are greater for higher harmonic components of measured signal. Nevertheless their influence on the total error is limited, because amplitudes of higher harmonics decrease significantly.

Result of simulations for exemplary linear systems

The method of the measurement described above was also tested in the case of two typical linear systems, which transfer functions were given by following formulas:

(20) $$H_1(s) = \frac{1}{0{,}05 \cdot f_0^{-1} \cdot s + 1},$$

(21) $$H_2(s) = \frac{1}{5 \cdot 10^{-4} \cdot f_0^{-2} \cdot s^2 + 0{,}01 \cdot f_0^{-1} \cdot s + 1}.$$

Their impulse responses are presented on Fig. 4.

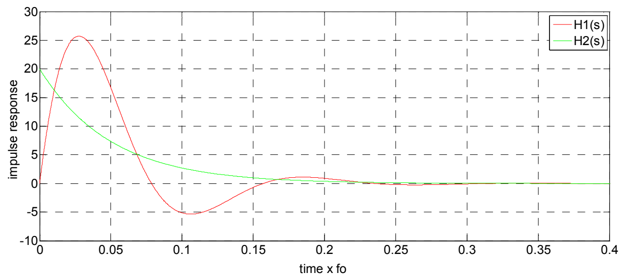

Fig. 4. Impulse responses of considered linear systems

In these cases errors of simulations δ_2 are defined as

$$\delta_2 = \frac{\sum_{p=1}^{P}\sum_{k=0}^{N-1}\left|\left[\bar{x}(k) - h_0\left(k \cdot f_0^{-1} \cdot N^{-1}\right)\right]\right|}{h_{0\max} \cdot P \cdot N}, \tag{22}$$

where h_{0max} is the maximum value of the impulse response. Results of the calculations, performed for identical conditions, as in the previous section, are presented on Figs. 5 – 8.

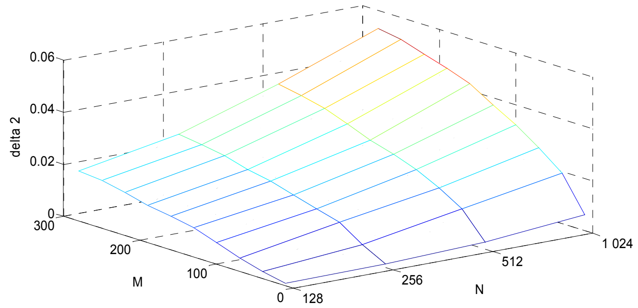

Fig. 5. A dependence of δ_2 on N and M for $H_1(s)$, $\delta_f=\delta_{Td}=10^{-6}$

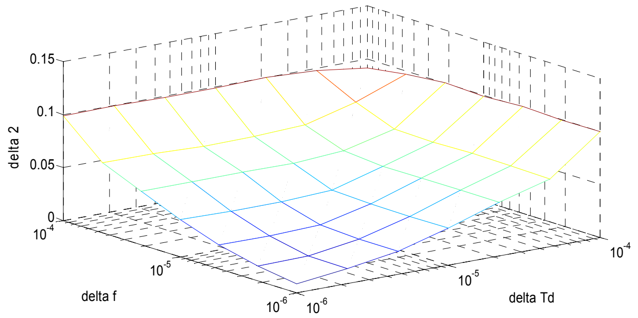

Fig. 6. A dependence of δ_2 on δ_f and δ_{Td} for $H_1(s)$, M=30 N=256

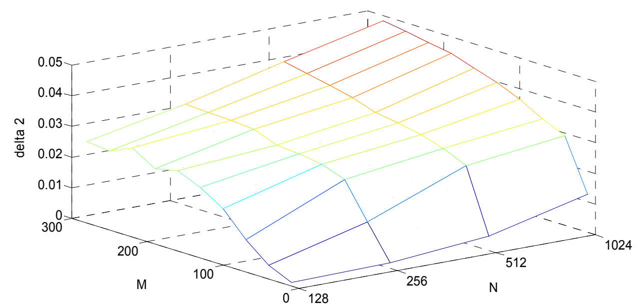

Fig. 7. A dependence of δ_2 on N and M for $H_2(s)$, $\delta_f=\delta_{Td}=10^{-6}$

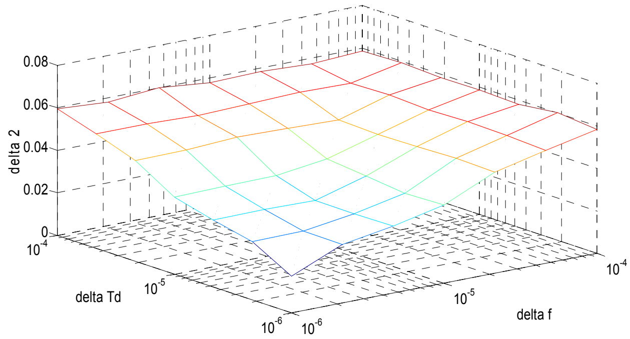

Fig. 8. A dependence of δ_2 on δ_f and δ_{Td} for $H_2(s)$, M=50 N=256

Conclusions

Presented results of calculations prove, that ΣΔ undersampling is the effective method of the measurement of the linear system's impulse response. The whole process is realized by means of the set of analogue integrators and simple digital filters. The method can be especially suitable in the cases, when parameters of the system require high sampling frequency, too high to apply sampling of the signal according to Shannon's theorem. The acceptable accuracy of the measurement is possible to achieve for fluctuations of sampling and signal's frequencies in a range of 10^{-6}. In this case the measurement can be performed with the error less than 1 % for M in the range of 100 and N = 256 or even 512.

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