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Minimal–phase positive electrical circuits

***Abstract**. Minimal-phase positive continuous-time linear electrical circuits are addressed. It is shown that positive asymptotically stable electrical circuits with distinct poles and zeros are minimal-phase systems. Conditions are established for electrical circuits to be minimal-phase systems. Sufficient conditions for cancelation of zeros and poles of minimal-phase electrical circuits are proposed.*

***Streszczenie.** W pracy są analizowane dodatnie minimalnofazowe obwody elektryczne opisane równaniami stanu i macierzami transmitancji operatorowych. Wykazano, że dodatnie stabilne asymptotycznie obwody elektryczne z różnymi zerami i biegunami są obwodami minimalnofazowymi. Podano warunki minimalnofazowości obwodów elektrycznych, oraz warunki wystarczające upraszczania zer i biegunów w obwodach elektrycznych. Rozważania ogólne zilustrowano przykładami obwodów elektrycznych. (**Dodatnie minimalnofazowe obwody elektryczne**).*

Keywords: minimal-phase, positive, asymptotically stable, electrical circuit, pole, zero, cancelation.
Słowa kluczowe: minimalnofazowość, dodatniość, stabilność asymptotyczna, obwody elektryczne, zera, bieguny.

Introduction

In electrical circuits the state variables and outputs take only non-negative values for any non-negative initial conditions and inputs. The positive standard and fractional order electrical circuits have been investigated in many papers and books [1-7]. A new class of normal electrical circuits has been introduced in [8]. The minimum energy control of electrical circuits has been investigated in [9]. Positive linear systems consisting of n subsystems with different fractional orders have been addressed in [10, 11]. Decoupling zeros of positive linear systems have been introduced in [12].

Determination of the state space equations for given transfer matrices is a classical problem, called the realization problem, which has been addressed in many papers and books [13-17]. An overview of the positive realization problem is given in [13, 14, 16, 18]. The realization problem for positive continuous-time and discrete-time linear system has been considered in [16, 19-27] and for linear systems with delays in [16, 19, 24, 27-29]. The realization problem for fractional linear systems has been analyzed in [16, 30-36] and for positive 2D hydrid linear systems in [29]. A new modified state variable diagram method for determination of positive realizations with reduced number of delays for given proper transfer matrices has been proposed in [37].

In this paper the minimal-phase realization problem and minimal-phase positive electrical circuits will be investigated.

The paper is organized as follows. In section 2 some preliminaries on positivity and asymptotic stability of continuous-time linear systems are recalled. Some definitions, theorems and examples of positive electrical circuits are presented in section 3. Main results of the paper are given in sections 4 and 5. Minimal-phase realization problem of continuous-time linear systems is discussed in section 4 and minimal-phase positive electrical circuits are investigated in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries, $\Re^{n\times m}(s)$ - the set of $n\times m$ rational matrices in s with real coefficients, I_n - the $n\times n$ identity matrix.

Preliminaries

Consider the continuous-time linear system

(1a) $\dot{x} = Ax + Bu$,

(1b) $y = Cx + Du$,

where $x\in\Re^n$, $u\in\Re^m$, $y\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 1. [18] The system (1) is called (internally) positive if $x = x(t)\in\Re_+^n$ and $y = y(t)\in\Re_+^p$, $t\in[0,+\infty]$ for all $x_0 = x(0)\in\Re_+^n$ and $u = u(t)\in\Re_+^m$, $t\in[0,+\infty]$.

Theorem 1. [18] The system (1) is positive if and only if

(2) $A\in M_n$, $B\in\Re_+^{n\times m}$, $C\in\Re_+^{p\times n}$, $D\in\Re_+^{p\times m}$,

where M_n is the set of $n\times n$ Metzler matrices, i.e. the matrices with nonnegative off-diagonal entries.

The transfer matrix of (1) is given by

(3) $$T(s) = C[I_n s - A]^{-1}B + D = \frac{N(s)}{d(s)} \in \Re^{p\times m}(s),$$

where $N(s)$ is the polynomial matrix and $d(s)$ is the polynomial.

For single-input single-output (SISO, $m = p = 1$) linear system the transfer function can be written in the form

(4) $$T(s) = \frac{n(s)}{d(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0}.$$

Definition 2. The roots s_1, s_2,…, s_n of the equation

(5) $$d(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 = (s - s_1)(s - s_2)...(s - s_n) = 0$$

are called the poles of the linear system.

Definition 3. The roots s_1^0, s_2^0,…, s_n^0 of the equation

(6) $$n(s) = b_n s^n + b_{n-1}s^{n-1} + ... + b_1 s + b_0 = b_n(s - s_1^0)(s - s_2^0)...(s - s_n^0) = 0$$

are called the zeros of the linear system.

The poles s_1, s_2,…, s_n and the zeros s_1^0, s_2^0,…, s_n^0 are called distinct if $s_i \neq s_j$ for $i\neq j$ and $s_i^0 \neq s_j^0$ for $i \neq j$, $i, j = 1,...,n$, respectively.

Definition 4. The linear system is called minimal-phase if

(7) $\operatorname{Re} s_k < 0$ and $\operatorname{Re} s_k^0 < 0$ for $k = 1,...,n$,

where Re denotes the real part of the complex number.

Definition 5. [9] The positive system (1) is called asymptotically stable if

(8) $\lim_{t\to\infty} x(t) = 0$ for all $x_0 \in \Re_+^n$.

Theorem 2. [18] The positive system (1) is asymptotically stable if and only if

(9) $\operatorname{Re} \lambda_k < 0$ for $k = 1,...,n$,

where λ_k is the eigenvalue of the matrix $A \in M_n$ and

(10) $\det[I_n\lambda - A] = (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n)$.

Note that the set of poles $\{s_1, s_2, \ldots, s_n\}$ in general case is the subset of the set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ [15].

Definition 6. The matrices *A*, *B*, *C*, *D* satisfying (2) are called a positive realization of a given transfer matrix $T(s)$ if they fulfill the equality (3).

Positive electrical circuits

Consider the linear continuous-time electrical circuit described by the state equations

(11a) $\dot{x}(t) = Ax(t) + Bu(t)$,

(11b) $y(t) = Cx(t) + Du(t)$,

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

It is well-known [3] that any linear electrical circuit composed of resistors, coils, capacitors and voltage (current) sources can be described by the state equations (11). Usually as the state variables $x_1(t), \ldots, x_n(t)$ (the components of the state vector $x(t)$) the currents in the coils and voltages on the capacitors are chosen.

Definition 7. [3] The electrical circuit (11) is called (internally) positive if $x(t) \in \Re_+^n$ and $y = y(t) \in \Re_+^p$, $t \in [0,+\infty]$ for any $x_0 = x(0) \in \Re_+^n$ and every $u(t) \in \Re_+^m$, $t \in [0,+\infty]$.

Theorem 3. [3] The electrical circuit (11) is positive if and only if

(12) $A \in M_n$, $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$, $D \in \Re_+^{p\times m}$.

Theorem 4. The linear electrical circuit composed of resistors, coils and voltage sources is positive for any values of the resistances, inductances and source voltages if the number of coils is less or equal to the number of its linearly independent meshes and the direction of the mesh currents are consistent with the directions of the mesh source voltages.

Proof. Proof is given in [3].

Theorem 5. The linear electrical circuit composed of resistors, capacitors and voltage sources is not positive for all values of its resistances, capacitances and source voltages if each its branch contains resistor, capacitor and voltage source.

Proof. Proof is given in [3].

Theorem 6. The electrical circuit shown in Figure 1 is positive for any values of the conductances G_k, $k = 0,1,...,n$; capacitances C_j, $j = 1,...,n$ and source voltage *e*.

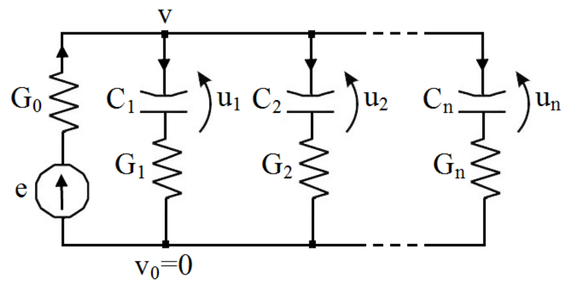

Fig. 1. Positive electrical circuit.

Proof. Proof is given in [3].

Theorem 7. The *R*, *L*, *C*, *e* electrical circuits are not positive for any values of its resistances, inductances, capacitances and source voltages if at least one its branch contains coil and capacitor.

Proof. Proof is given in [3].

Theorem 8. The linear electrical circuit of the structure shown in Figure 2 is positive for any values of its resistances R_k, $k = 1,2,...,n$, inductances L_k, $k = 2,4,...,n_2$ and capacitances C_k, $k = 1,3,...,n_1$.

Proof. Using Kirchhoff's laws we can write the equations

(13a) $e_0 = R_k C_k \frac{du_k}{dt} + u_k$, $k = 1,3,...,n_1$,

(13b) $e_0 + e_j = L_j \frac{di_j}{dt} + R_j i_j$, $j = 2,4,...,n_2$,

which can be written in the form

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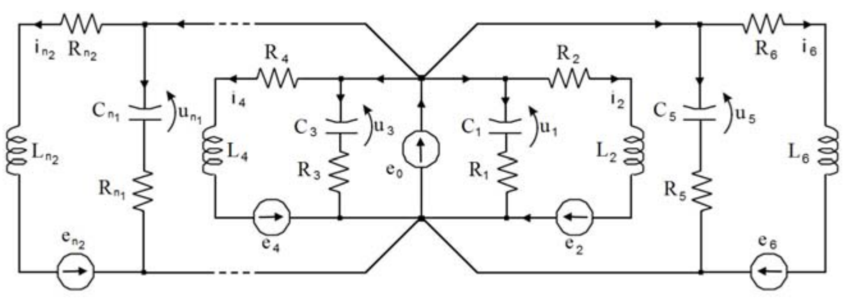

Fig. 2. Positive electrical circuit.

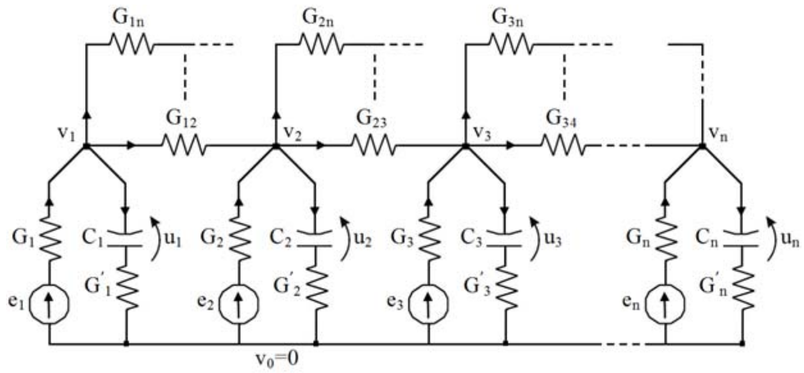

Fig. 3. Positive electrical circuit.

(14a) $$\frac{d}{dt}\begin{bmatrix} u \\ i \end{bmatrix} = A\begin{bmatrix} u \\ i \end{bmatrix} + Be,$$

where

(14b) $$u = \begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_{n_1} \end{bmatrix},\ i = \begin{bmatrix} i_2 \\ i_4 \\ \vdots \\ i_{n_2} \end{bmatrix},\ e = \begin{bmatrix} e_0 \\ e_2 \\ e_4 \\ \vdots \\ e_{n_2} \end{bmatrix}$$

and

(14c)

$$A = \mathrm{diag}\begin{bmatrix} \frac{1}{R_1C_1} & \frac{1}{R_3C_3} & \cdots & \frac{1}{R_{n_1}C_{n_1}} & \frac{R_2}{L_2} & \frac{R_4}{L_4} & \cdots & \frac{R_{n_2}}{L_{n_2}} \end{bmatrix} \in M_n,$$

$$B_1 = \begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 & \cdots & 0 \\ \frac{1}{R_3C_3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{1}{R_{n_1}C_{n_1}} & 0 & 0 & \cdots & 0 \end{bmatrix} \in \Re_+^{n_1\times m},\ B_2 = \begin{bmatrix} \frac{1}{L_2} & \frac{1}{L_2} & 0 & \cdots & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_{n_2}} & 0 & 0 & \cdots & \frac{1}{L_{n_2}} \end{bmatrix} \in \Re_+^{n_2\times m},$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

From (14c) it follows that the electrical circuit is positive for any values of its resistances R_k, $k = 1,2,...,n$, inductances L_k, $k = 2,4,...,n_2$ and capacitances C_k, $k = 1,3,...,n_1$. □

Theorem 9. The linear electrical circuit of the structure shown in Figure 3 is positive for any values of its conductances G_k, $G_k^{'}$, G_{kj}, $k, j = 1,...,n$, capacitances C_k, $k = 1,...,n$ and source voltages e_k, $k = 1,...,n$.

Proof. Proof is given in [3].

The state equations for the positive electrical circuit shown in Figure 3 are given in [3].

Theorem 10. The positive electrical circuit G, C, i_s type is unstable if it has at least one node with branches containing only capacitors and current sources.

Proof. Proof is given in [3].

Theorem 11. The positive electrical circuit R, L, C, e type is unstable if it has at least one mesh containing only the inductances and source voltages.

Proof. Proof is given in [3].

Positive minimal-phase realizations of continuous-time linear systems

First let us consider the SISO continuous-time linear system with the transfer function (4). From (4) we have

(16) $$D = \lim_{s\to\infty} T(s) = b_n$$

and the strictly proper transfer function has the form

(17a) $$T_{sp}(s) = T(s) - D = C[I_n s - A]^{-1}B = \frac{\hat{b}_{n-1}s^{n-1} + ... + \hat{b}_1 s + \hat{b}_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0} = \frac{\hat{n}(s)}{d(s)},$$

where

(17b) $\hat{b}_k = b_k - b_n a_k$, $k = 0,1,...,n-1$,

(17c) $\hat{n}(s) = \hat{b}_{n-1}s^{n-1} + ... + \hat{b}_1 s + \hat{b}_0$.

It is assumed that the poles s_1, s_2,…, s_n and the zeros s_1^0, s_2^0,…, s_{n-1}^0 of (17) are distinct, real, negative and satisfy the conditions

(18) $s_k \le s_k^0 \le s_{k+1}$ for $k = 1,...,n-1$.

It is well-known [16] that the strictly proper transfer function (17) can be written in the form

(19a) $$T_{sp}(s) = \sum_{k=1}^{n} \frac{T_k}{s - s_k},$$

where

(19b) $$T_k = \lim_{s\to s_k}(s - s_k)T_{sp}(s) = \frac{\hat{n}(s_k)}{\prod_{\substack{j=1 \\ j\ne k}}^{n}(s_k - s_j)}.$$

Note that $T_k > 0$ for $k = 1,...,n$ if and only if the poles and zeros are distinct and satisfy the condition (18). In this case we can choose $c_k > 0$, $b_k > 0$ so that

(20) $T_k = c_k b_k$, $k = 1,...,n$

and the matrices

$$A = \mathrm{diag}[s_1 \quad s_2 \quad \cdots \quad s_n] \in M_n,\ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \Re_+^{n\times 1},$$

(21) $C = [c_1 \quad c_2 \quad \cdots \quad c_n] \in \Re_+^{1\times n}$

are a positive realization of the transfer function (17).

Theorem 12. There exists minimal-phase realization (21), (16) of the transfer function (4) if and only if the poles and zeros of (17) are distinct, real, negative and the conditions (18) are satisfied.

The proof and procedure for computation of the realization are given in [38].

Now let us consider the *m*-inputs and *p*-outputs (MIMO) continuous-time linear system with the strictly proper transfer matrix

(22a) $$T_{sp}(s) = \frac{N(s)}{d(s)} \in \Re^{p\times m}(s),$$

where

(22b) $$d(s) = (s-s_1)(s-s_2)...(s-s_n),$$

(22c) $$N(s) = \begin{bmatrix} (s-s_{11}^{0,1})...(s-s_{11}^{0,n_{11}}) & \cdots & (s-s_{1m}^{0,1})...(s-s_{1m}^{0,n_{1m}}) \\ \vdots & \ddots & \vdots \\ (s-s_{p1}^{0,1})...(s-s_{p1}^{0,n_{p1}}) & \cdots & (s-s_{pm}^{0,1})...(s-s_{pm}^{0,n_{pm}}) \end{bmatrix}$$

with distinct real negative poles $s_1, s_2, ..., s_n$ and distinct real negative zeros $s_{11}^{0,1}, ..., s_{11}^{0,n_{11}}, s_{1m}^{0,1}, ..., s_{pm}^{0,n_{pm}}$.

The transfer matrix (22) can be written in the form

(23a) $$T_{sp}(s) = \sum_{k=1}^{n} \frac{T_k}{s-s_k},$$

where

(23b) $$T_k = \lim_{s\to s_k}(s-s_k)T_{sp}(s) = \frac{N(s_k)}{\prod_{\substack{j=1 \\ j\neq k}}^{n}(s_k - s_j)}$$

and

(24) $$\operatorname{rank} T_k = r_k \le \min(m, p).$$

It is easy to check that if the conditions

(25) $$s_k \le s_{ij}^{0,k} \le s_{k+1} \text{ for } i = 1,...,p,\ j = 1,...,m,\ k = 1,...,n_{ij}$$

are satisfied then $T_k \in \Re_+^{p\times m}$ for $k = 1,...,n$ and it can be written as the product

(26a) $$T_k = C_k B_k,$$

where

$$C_k \in \Re_+^{p\times r_k},\ B_k \in \Re_+^{r_k\times m}$$

(26b) and $\operatorname{rank} C_k = \operatorname{rank} B_k = r_k$, $k = 1,...,n$.

It can be shown that the matrices

$$A = \operatorname{blockdiag}[I_{r_1}s_1 \quad \cdots \quad I_{r_n}s_n] \in M_r,$$

(27) $$B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \in \Re_+^{r\times m},\ C = [C_1 \quad \cdots \quad C_n] \in \Re_+^{p\times r},\ r = \sum_{i=1}^{n} r_i$$

are a positive realization of the matrix (22).

Theorem 13. There exists a minimal-phase realization (27) of the strictly proper transfer matrix (22) if and only if the poles and zeros are distinct, real, negative and the conditions (24) are satisfied.

Minimal-phase positive electrical circuits

First we shall show the essence of the approach on simple examples of positive electrical circuits.

Example 1. Consider the positive electrical circuit shown in Figure 4 with positive resistances R_1, R_2, R_3, capacitances C_1, C_2 and source voltage *e*. As the state variables we choose the voltages u_1, u_2 on the capacitors and as the output *y* their sum.

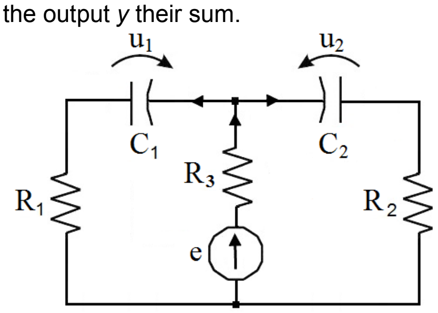

Fig. 4. Positive electrical circuit of Example 1.

Using Kirchhoff's laws we may write the equations

(28a) $$e = R_1C_1\frac{du_1}{dt} + u_1 + R_3\left(C_1\frac{du_1}{dt} + C_2\frac{du_2}{dt}\right),$$

(28b) $$e = R_2C_2\frac{du_2}{dt} + u_2 + R_3\left(C_1\frac{du_1}{dt} + C_2\frac{du_2}{dt}\right)$$

and

(29) $$y = u_1 + u_2.$$

The equations (28) and (29) can be rewritten in the form

(30a) $$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Be,$$

(30b) $$y = C\begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\frac{R_2+R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} & \frac{R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \frac{R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} & -\frac{R_1+R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix}$$

(30c)

$$B = \begin{bmatrix} \frac{R_2}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \frac{R_1}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix},\ C = [1 \quad 1].$$

The transfer function of the electrical circuit has the form

$$T(s) = C[I_2s - A]^{-1}B = [1 \quad 1]$$

$$\times\begin{bmatrix} s + \frac{R_2+R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} & \frac{-R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \frac{-R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} & s + \frac{R_1+R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix}^{-1}$$

(31a) $\times\begin{bmatrix} \dfrac{R_2}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \dfrac{R_1}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix} = \dfrac{n(s)}{d(s)},$

where

(31b) $n(s) = (C_1R_1 + C_2R_2)s + 2$,

(31c) $$d(s) = C_1C_2[R_1(R_2+R_3)+R_2R_3]s^2 + [C_1(R_1+R_3)+C_2(R_2+R_3)]s + 1 = as^2 + bs + 1.$$

The poles of the electrical circuit are

(32) $$s_1 = \frac{-b+\sqrt{b^2-4a}}{2a}, \; s_2 = \frac{-b-\sqrt{b^2-4a}}{2a}$$

and its zero is

(33) $$s_1^0 = -\frac{2}{C_1R_1 + C_2R_2}.$$

It is easy to see that the poles (32) and zero (33) satisfy the condition (17) for any positive resistances R_1, R_2, R_3 and capacitances C_1, C_2 since always $b^2 - 4a \geq 0$.

Example 2. Consider the positive electrical circuit shown in Figure 5 with positive resistances R_1, R_2, R_3, inductances L_1, L_2 and source voltages e_1, e_2. As the state variables we choose the currents i_1, i_2 in the coils and as the output y the voltages on the resistances R_1, R_2.

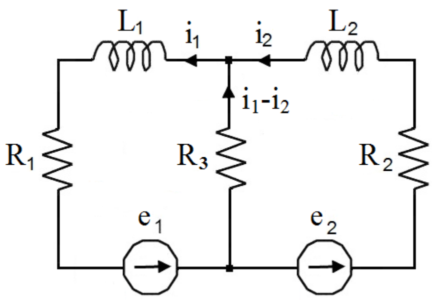

Fig. 5. Positive electrical circuit of Example 2.

Using Kirchhoff's laws we may write the equations

(34a) $$e_1 = R_1i_1 + L_1\frac{di_1}{dt} + R_3(i_1 - i_2),$$

(34b) $$e_2 = R_2i_2 + L_2\frac{di_2}{dt} + R_3(i_2 - i_1)$$

and

(35) $$y = \begin{bmatrix} R_1i_1 \\ R_2i_2 \end{bmatrix}.$$

The equations (34) and (35) can be rewritten in the form

(36a) $$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

(36b) $$y = C\begin{bmatrix} i_1 \\ i_2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\dfrac{R_1+R_3}{L_1} & \dfrac{R_3}{L_1} \\ \dfrac{R_3}{L_2} & -\dfrac{R_2+R_3}{L_2} \end{bmatrix}, \; B = \begin{bmatrix} \dfrac{1}{L_1} & 0 \\ 0 & \dfrac{1}{L_2} \end{bmatrix},$$

(36c) $$C = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}.$$

The transfer matrix of the electrical circuit has the form

$$T(s) = C[I_2s - A]^{-1}B = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

(37a) $$\times\begin{bmatrix} s+\dfrac{R_1+R_3}{L_1} & -\dfrac{R_3}{L_1} \\ -\dfrac{R_3}{L_2} & s+\dfrac{R_2+R_3}{L_2} \end{bmatrix}^{-1}\begin{bmatrix} \dfrac{1}{L_1} & 0 \\ 0 & \dfrac{1}{L_2} \end{bmatrix} = \frac{N(s)}{d(s)},$$

where

(37b) $$N(s) = \begin{bmatrix} R_1(R_2+R_3+sL_2) & R_1R_3 \\ R_2R_3 & R_2(R_1+R_3+sL_1) \end{bmatrix},$$

(37c) $$d(s) = L_1L_2s^2 + [L_1(R_2+R_3)+L_2(R_1+R_3)]s + R_1(R_2+R_3)+R_2R_3 = as^2 + bs + c.$$

The poles of the electrical circuit are

(38) $$s_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \; s_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

and its zeros are

(39) $$s_1^0 = \frac{-(R_2+R_3)}{L_2}, \; s_2^0 = \frac{-(R_1+R_3)}{L_1}.$$

It is easy to see that the poles (38) and zeros (39) satisfy the condition (17) for any positive resistances R_1, R_2, R_3 and inductances L_1, L_2 since always $b^2 - 4ac \geq 0$.

Example 3. Consider the positive electrical circuit shown in Figure 6 with given positive resistances R_1, R_2, R, inductance L, capacitances C_1, C_2 and source voltage e.

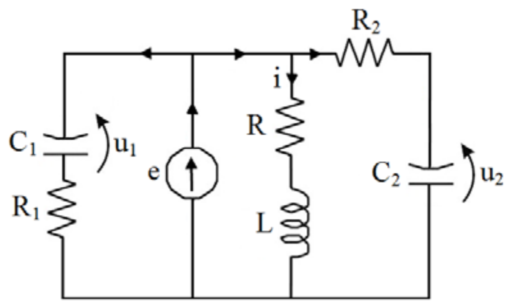

Fig. 6. Positive electrical circuit of Example 3.

Using Kirchhoff's laws we may write the equations

(40a) $$e = R_1C_1\frac{du_1}{dt} + u_1,$$

(40b) $$e = Ri + L\frac{di}{dt},$$

(40c) $$e = R_2C_2\frac{du_2}{dt} + u_2,$$

which can be written in the form

(41a) $$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ i \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_2 \\ i \end{bmatrix} + Be,$$

where

(41b) $$A = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 & 0 \\ 0 & -\frac{1}{R_2C_2} & 0 \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \\ \frac{1}{L} \end{bmatrix}.$$

As the output *y* we choose

(42) $$y = u_1 + Ri = C\begin{bmatrix} u_1 \\ u_2 \\ i \end{bmatrix}, \quad C = [1 \quad 0 \quad R].$$

The transfer function of the electrical circuit has the form

(43)
$$T(s) = C[I_3 s - A]^{-1}B$$
$$= [1 \quad 0 \quad R]\begin{bmatrix} s+\frac{1}{R_1C_1} & 0 & 0 \\ 0 & s+\frac{1}{R_2C_2} & 0 \\ 0 & 0 & s+\frac{R}{L} \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \\ \frac{1}{L} \end{bmatrix}$$
$$= [1 \quad 0 \quad R]\frac{1}{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)\left(s+\frac{R}{L}\right)}$$
$$\times\begin{bmatrix} \left(s+\frac{1}{R_2C_2}\right)\left(s+\frac{R}{L}\right) & 0 & 0 \\ 0 & \left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{R}{L}\right) & 0 \\ 0 & 0 & \left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right) \end{bmatrix}$$
$$\times\begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \\ \frac{1}{L} \end{bmatrix} = \frac{n(s)}{d(s)},$$

where

(44)
$$n(s) = \left(s+\frac{1}{R_2C_2}\right)\left(s+\frac{R}{L}\right)\frac{1}{R_1C_1} + \left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)\frac{R}{L}$$
$$= \left(s+\frac{1}{R_2C_2}\right)\left[\left(s+\frac{R}{L}\right)\frac{1}{R_1C_1} + \left(s+\frac{1}{R_1C_1}\right)\frac{R}{L}\right],$$

(45) $$d(s) = \left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)\left(s+\frac{R}{L}\right).$$

The poles of the electrical circuit are

(46) $$s_1 = -\frac{1}{R_1C_1}, \; s_2 = -\frac{1}{R_2C_2}, \; s_3 = -\frac{R}{L}$$

and its zeros are

(47) $$s_1^0 = -\frac{1}{R_2C_2}, \; s_2^0 = -\frac{2R}{RR_1C_1}.$$

If $R_1C_1 \geq R_2C_2$ and $\frac{R}{L} \geq \frac{1}{R_2C_2}$, then the poles and zeros satisfy the condition (17).
Therefore, the positive electrical circuit is asymptotically stable and minimal-phase.
Note that the zero s_1^0 is equal to the pole s_2 since the matrix A is diagonal and after the cancelation of the zero and pole the transfer function has the form

(48) $$T(s) = \frac{\left(\frac{1}{R_1C_1}+\frac{R}{L}\right)s + \frac{2R}{R_1C_1L}}{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{R}{L}\right)}.$$

In general case we have the following theorem.
Theorem 14. If $A = \mathrm{diag}[-a_1 \quad -a_2 \quad \cdots \quad -a_n] \in M_n$ and at least one entry in the matrix $B = [b_1 \quad b_2 \quad \cdots \quad b_n]^T \in \Re_+^n$ or in the matrix $C = [c_1 \quad c_2 \quad \cdots \quad c_n] \in \Re_+^{1\times n}$ is zero, then at least one zero of the electrical circuit is equal to one of its poles.
Proof. Let $c_2 = 0$, then the transfer function of the electrical circuit has the form

(49)
$$T(s) = C[I_n s - A]^{-1}B = [c_1 \quad 0 \quad c_3 \quad \cdots \quad c_n]$$
$$\times[\mathrm{diag}(s+a_1 \quad s+a_2 \quad \cdots \quad s+a_n)]^{-1}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$
$$= [c_1 \quad 0 \quad c_3 \quad \cdots \quad c_n]\frac{1}{(s+a_1)(s+a_2)...(s+a_n)}$$
$$\times\mathrm{diag}[(s+a_2)(s+a_3)...(s+a_n) \quad (s+a_1)(s+a_3)...(s+a_n) \quad ...$$
$$\cdots \quad (s+a_1)(s+a_2)...(s+a_{n-1})]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$
$$= \frac{(s+a_2)[c_1b_1(s+a_3)...(s+a_n) + c_nb_n(s+a_1)(s+a_3)...(s+a_{n-1})]}{(s+a_1)(s+a_2)...(s+a_n)}$$
$$= \frac{c_1b_1(s+a_3)...(s+a_n) + c_3b_3(s+a_1)(s+a_4)...(s+a_{n-1})}{(s+a_1)(s+a_3)...(s+a_n)}.$$

Therefore, the pole $s_2 = -a_2$ is also the zero of the electrical circuit. The proof if one entry of the matrix *B* is zero is similar. □
Theorem 14 can be easily extended to MIMO positive asymptotically stable electrical circuits.

Example 4. Consider the positive electrical circuit shown in Figure 2 for $n_1 = 3$, $n_2 = 4$ with given positive resistances R_1, R_2, R_3, R_4, inductances L_2, L_4, capacitances C_1, C_3 and source voltages e_0, e_2, e_4. In this case the state equations have the form

(50a) $$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_3 \\ i_2 \\ i_4 \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_3 \\ i_2 \\ i_4 \end{bmatrix} + B\begin{bmatrix} e_0 \\ e_2 \\ e_4 \end{bmatrix},$$

where

$$A = \mathrm{diag}\left[-\frac{1}{R_1C_1} \quad -\frac{1}{R_3C_3} \quad -\frac{R_2}{L_2} \quad -\frac{R_4}{L_4}\right],$$

(50b) $$B = \begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \\ \frac{1}{R_3C_3} & 0 & 0 \\ \frac{1}{L_2} & \frac{1}{L_2} & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} \end{bmatrix}.$$

As the output of the electrical circuit we choose

(51) $$y = u_3 + i_4 = C\begin{bmatrix} u_1 \\ u_3 \\ i_2 \\ i_4 \end{bmatrix}, \; C = [0 \quad 1 \quad 0 \quad 1].$$

The transfer matrix of the electrical circuit has the form

(52)

$$T(s) = C[I_4s - A]^{-1}B = [0 \;\; 1 \;\; 0 \;\; 1]$$
$$\times\left\{\mathrm{diag}\left[\left(s+\frac{1}{R_1C_1}\right),\left(s+\frac{1}{R_3C_3}\right),\left(s+\frac{R_2}{L_2}\right),\left(s+\frac{R_4}{L_4}\right)\right]\right\}^{-1}$$
$$\times\begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \\ \frac{1}{R_3C_3} & 0 & 0 \\ \frac{1}{L_2} & \frac{1}{L_2} & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} \end{bmatrix} = \frac{1}{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_3C_3}\right)\left(s+\frac{R_2}{L_2}\right)\left(s+\frac{R_4}{L_4}\right)}$$
$$\times\left[0 \quad \left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{R_2}{L_2}\right)\left(s+\frac{R_4}{L_4}\right) \quad 0 \quad \left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_3C_3}\right)\left(s+\frac{R_2}{L_2}\right)\right]$$
$$\times\begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \\ \frac{1}{R_3C_3} & 0 & 0 \\ \frac{1}{L_2} & \frac{1}{L_2} & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} \end{bmatrix} = \left[\frac{1}{R_3C_3s+1} + \frac{1}{L_4s+R_4} \quad 0 \quad \frac{1}{L_4s+R_4}\right].$$

From (52) it follows that in this case three zeros of the electrical circuit are equal to the corresponding poles.

Theorem 15. In SISO positive asymptotically stable electrical circuits the distinct negative zeros s_k^0, $k = 1,...,n$ and the distinct negative poles s_j, $j = 1,...,n$ satisfy the condition (17).

Proof. The proof follows from Theorem 3.1. By this theorem there exists a minimal-phase realization (20) of (16) if and only if the poles and zeros are distinct and negative and satisfy the conditions (17). □

Theorem 15 can be easily extended to MIMO positive asymptotically stable electrical circuits.

Theorem 16. In MIMO positive asymptotically stable electrical circuits the distinct negative zeros $s_{ij}^{0,k}$, $i = 1,..., p$, $j = 1,...,m$, $k = 1,...,n_{ij}$ and the distinct negative poles s_k, $k = 1,...,n$ satisfy the conditions (24).

Concluding remarks

Minimal-phase positive electrical circuits has been addressed. The minimal-phase realization problem for positive electrical circuits has been analyzed. I has been shown that the positive asymptotically stable electrical circuits with distinct poles and zeros are minimal-phase and satisfy the conditions (24) (Theorems 15 and 16). Sufficient conditions for cancelation of zeros and poles of minimal-phase electrical circuits have been established (Theorem 14). The considerations have been illustrated by examples of positive minimal-phase electrical circuits. The presented results can be extended to fractional order positive electrical circuits.

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