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doi:10.15199/48.2016.07.23

The procedure of construction of mathematical models for nonlinear dynamical systems based on optimization approach

Abstract. *Using optimization for mathematical models' construction is a universal approach that can be applied to a wide set of objects. Efficient application of this approach to nonlinear dynamical objects requires a combination of methods to be used in order to obtain a good model and to reasonably limit the amount of required computations. An overview of such methods with the example of a complex model construction is provided in this paper.*

Streszczenie. *Wykorzystanie optymalizacji w tworzeniu modeli matematycznych obiektów jest szeroko stosowanym podejściem. Aby zapewnić wydajną optymalizację nieliniowych obiektów dynamicznych należy wykorzystać techniki upraszczające problem oraz przyspieszające obliczenia. W artykule zawarto przegląd takich metod, a następnie użyto ich do tworzenia złożonego modelu. (**Procedura tworzenia modeli matematycznych nieliniowych obiektów dynamicznych z wykorzystaniem metod optymalizacji**).*

Keywords: macromodels, optimization, parallelization, evolution.
Słowa kluczowe: makromodele, optymalizacja, metody zrównoleglania obliczeń, algorytmy ewolucyjne.

Introduction

The process of design and analysis of modern dynamical systems with large number of components of different nature requires significant computation resources. It is caused by considerable number of components of the system being designed and variety of physical phenomena to be taken into account. The usage of macromodels in such conditions allows us to significantly reduce the required computation efforts because it makes it possible to ignore unimportant effects for particular analysis. Macromodels can be used to describe single components as well as subsystems of significant size including elements of different nature. Such state of the problem to be considered leads to the necessity to develop universal approaches intended for construction of complex dynamical object macromodels in the form useful for their further analysis.

In a general case, the dynamical system for which the macromodel will be constructed can be represented as a multipole as shown schematically in Figure 1.

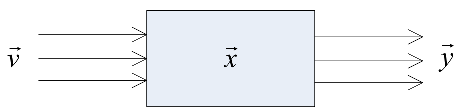

Fig. 1. An object for the macromodel construction

In Figure 1 vector $\vec{v}$ describes input values, vector $\vec{y}$ contains output values, vector $\vec{x}$ consists of values, corresponding to the internal state of the object. The goal of the research is to find an operator which will allow calculating output values $\vec{y}$ of the object using known input values $\vec{v}$ and initial values $\vec{x}$ of internal state of the object.

One of the most convenient forms of mathematical model representation comprises discrete state equations:

$$\begin{cases} \vec{x}^{(k+1)} = \mathbf{F}\vec{x}^{(k)} + \mathbf{G}\vec{v}^{(k)} + \Phi(\vec{x}^{(k)}, \vec{v}^{(k)}) \\ \vec{y}^{(k+1)} = \mathbf{C}\vec{x}^{(k+1)} + \mathbf{D}\vec{v}^{(k+1)} \end{cases} \tag{1}$$

where $\mathbf{F}, \mathbf{G}, \mathbf{C}, \mathbf{D}$ are matrices of the model coefficients; Φ is some nonlinear vector-function of many arguments; k is a discrete index.

The discrete form of representation is best suited for computer calculations because it allows omitting approximation of input data arrays. The state equations form is also convenient for further application of the model as a component of a dynamic system containing a greater number of elements.

One of the promising approaches for macromodel identification is the use of optimization. The idea of this approach is as follows:

The selected mathematical form of macromodel representation is described by a number of unknown parameters which we denote as vector $\vec{\lambda}$. For the case of discrete state equations (1) this vector includes elements of the matrixes $\mathbf{F}, \mathbf{G}, \mathbf{C}, \mathbf{D}$ and coefficients of the vector-function Φ.

Input information regarding the object can be presented as a set of transient characteristics $\left\{\vec{y}_i^{(k)}\right\}$ caused by input disturbances $\left\{\vec{v}_i^{(k)}\right\}$ where k is a discrete index, i is an index of transient characteristic.

Let us introduce some goal function representing a measure of inaccuracy with which our model describes the object. Its simplest mathematical representation can take a form of the root-mean square deviation:

$$Q(\vec{\lambda}) = \frac{1}{N}\sum_i \sum_k \left(\tilde{\vec{y}}_i^{(k)} - \vec{y}^{(k)}\right)^2 \tag{2}$$

where N is a total number of discrete values representing all transient characteristics, $\tilde{\vec{y}}_i^{(k)}$ is the object response calculated using its model.

The best possible values of macromodel coefficients correspond to such vector $\vec{\lambda}^*$ for which the goal function (2) approaches its minimum Thus, the model coefficients determination can be carried out by finding the global minimum point of the function (2).

The proposed approach is suitable for coefficients identification if the model is presented in any form described by a limited set of coefficients. Also, it does not apply any special restriction concerning the input information used for the coefficients identification. This makes it possible to use the proposed approach effectively as a universal method for coefficients identification during dynamical models construction.

The main disadvantage of the macromodel construction approach based on optimization is the complexity of corresponding optimization problem. The main factors which make this optimization problem complex are:

- high dimension of vector $\vec{\lambda}^*$ that is defined by the number of macromodel coefficients;
- strong nonlinear interrelation between the model coefficients;
- large number of operations required to calculate goal function;
- the presence of local minimums caused by both nonlinearity of the goal function and rounding errors;
- big difference between magnitudes of optimal coefficient values.

Thus many techniques were developed to simplify the optimization problem and accelerate its solution. In particular:

The division of the model identification process into stages. The effectiveness of this approach is caused by exponential dependency of computational complexity of the optimization problem on the number of its coefficients. So the division of the identification process into stages and corresponding reduction of the number of coefficients that are simultaneously optimized results in significant decrease of the total amount of calculations required. The most known algorithms for such division are:

The division by output variables, in which a separate sub-model is built for each output variable of the object (for each element of vector $\vec{y}$). All other each output variables of the object are considered as input variables (so they are added to vector $\vec{v}$). A final model is obtained from these sub-models using algebraic transformations.

The separation of linear sub-model. In this case the linear sub-model of the object is built first. The small signal mode of operation can be used for this. At the next stage a nonlinear model is built while model coefficients that belong to the linear part do not change. Finally, the general optimization of all model coefficients can be done.

Using alternative approaches to determine some model coefficients. For example, it is often possible to calculate elements of matrix $\mathbf{D}$ in equations (1) on the basis of zero data readout assuming that for $\vec{v}=0$ we have $\vec{x}=0$.

Adding a refining sub-model. This approach consists in building an additional model M2 to refine the obtained model M1 (see Fig. 2)

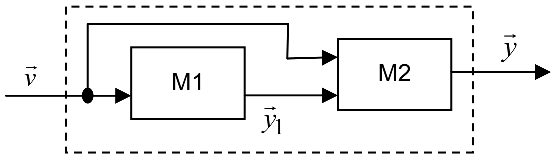

Fig. 2. Refined model is a combination of original model M1 and refining model M2

Performing an expert analysis to select an optimal model form. In this case, the analysis of the physical processes that take place in the object is performed and model representation is selected accordingly. As a result, the selected model form has much less coefficients in comparison to general form.

Scaling. Dynamical models being constructed are usually featured in practice by big difference between magnitudes of their coefficients. Thus the scaling has to be effective. Different approaches can be used to do this; in particular, there are automatic scaling algorithms. Moreover, one can scale the input and output values to be of the same value of magnitude.

Parallelizing of calculations. To accelerate the optimization problem solving it is advisable to parallelize the calculations. This allows us to perform the model identification faster by using more calculation resources.

Selection of effective optimization algorithm, that is optimal for particular optimization problem. It is known that our optimization problem has ravine-type dependencies and many local minima. Thus, it is recommended to use stochastic optimization algorithms. In particular, the authors used Rastrigin's cone method with adaptation of algorithm parameters.

Considering the variety of methods described above it can be stated that the most efficient model construction approach is a combination of them. To illustrate this, let's consider the construction of a model in instantaneous values based on experimental data simulating a single-phase asynchronous motor with the starting capacitor It should be noted, that several models were already built for the particular object [1, 2] using different approaches but those model have limited application domain because they were built using short transient characteristics that included a single transition process. As a result, the models obtained are unsuitable for simulation of processes different from those used during model construction. In particular, they incorrectly reflect the dependency of the object behaviour on its torque.

The construction of a simple and at the same time accurate model of the object that can simulate a wide range of regimes is described in this article. To achieve this goal the transient characteristics used for the model constructed have been obtained on the basis of three transition processes: the engine start, increase and decrease of the mechanical load. The applied voltage and mechanical load have been selected as input variables while current and rotor rotation speed have been selected as output variables. Fig. 3 and 4 present a general view of this data and its enlarged fragment at the time of mechanical load increase.

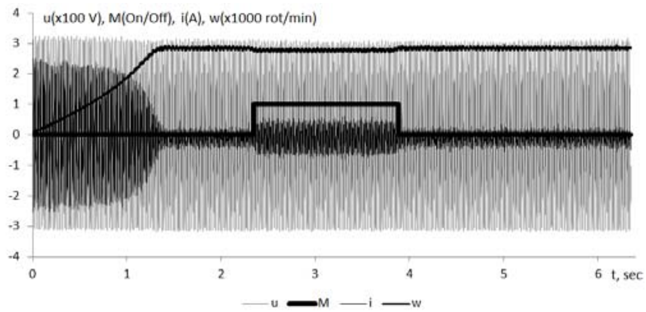

Fig. 3. The input data for the model construction

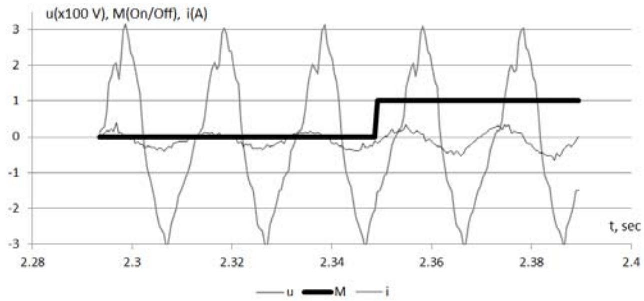

Fig. 4. Enlarged fragment of the input data for the model construction at the time of the mechanical load increase. Rotor rotation speed does not change within this fragment and thus it is not shown

Model construction

The model has been built in the form of discrete state equations (1) with nonlinearities approximated by cubic polynomial.

It should be noted that input data arrays include a large number of elements (about 13000). This results in a large amount of calculations required. About 10^6 floating point operations are needed to calculate a single value of the goal function. And because the optimization algorithm requires many (~10^2) goal function calculations at one iteration and the expected number of iterations is also significant (~10^5), the expected total number of floating point operations required is about 10^{13}. As a matter of fact, the required number of operations was even larger, in particular when performing evolutionary selection of the optimal model form.

In such situation it is reasonable to use parallelization of calculations. The authors have applied two types of parallelization: several CPU cores of one PC have been used to calculate goal function for different coefficient sets at one iteration of optimization algorithm and many PCs communicating via internet have been used to evaluate alternative model forms in evolutionary selection of the best one.

To simplify model construction and to obtain better result the model construction has been performed in several stages.

At the first stage the expert analysis was used. The model being built is a model of electrical engine. So we can use known dependencies:

$$\omega = \int (C_1 ui - C_2 M)dt \tag{3}$$

$$i = C_4 u - C_5 \omega u \tag{4}$$

where $C_i, i = \overline{1,5}$ are some coefficients.

It is obvious that the equations (3) and (4) do not describe the object precisely, but they represent the most important dependencies inside the object. Thus the corresponding model has the following form:

$$\begin{cases} x_1^{(k+1)} = G_{11}u^{(k)} + \alpha_1 x_2^{(k)} \cdot u^{(k)} \\ x_2^{(k+1)} = x_2^{(k)} + G_{22}M^{(k)} + \alpha_2 x_1^{(k)} u^{(k)} \\ i^{(k+1)} = x_1^{(k+1)} \\ \omega^{(k+1)} = x_2^{(k+1)} \end{cases} \tag{5}$$

This model includes only 4 unknown coefficients: G_{11}, G_{22}, α_1 and α_2, so its identification using optimization is not complex. The response of the obtained model to the test input is shown at Figure 5. The mean-root-square error of this model is of 14.4%.

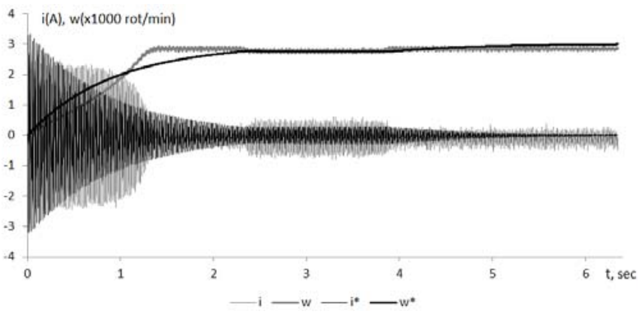

Fig. 5. Output variables of the object. Measured experimental values are shown in grey; values calculated using model (5) are shown in black

As it was expected, this model is not sufficiently precise. The reason consists in its too simple form. Nevertheless, it reflects the most fundamental dependencies inherent to engines.

To improve the model accuracy the evolutionary selection of the model form was used at the second stage. The search was done within the cubic approximation with two components of the state vector. The model (5) was taken as an initial approximation.

Much bigger amount of computations was needed at this stage. The total computation time was about 10 hours with 4-core PC. The obtained model has the following form:

$$\begin{cases} x_1^{(k+1)} = G_{11}u^{(k)} + \alpha_1 x_2^{(k)} u^{(k)} + \\ \quad + \beta_1 \left(x_1^{(k)}\right)^3 + \beta_0 \left(x_2^{(k)}\right)^3 \\ x_2^{(k+1)} = F_{22}x_2^{(k)} + G_{22}M^{(k)} + \alpha_2 x_1^{(k)} u^{(k)} + \\ \quad + \beta_2 \left(x_1^{(k)}\right)^2 x_2^{(k)} + \beta_3 x_1^{(k)} x_2^{(k)} u^{(k)} \\ i^{(k+1)} = x_1^{(k+1)} \\ \omega^{(k+1)} = x_2^{(k+1)} \end{cases} \tag{6}$$

The corresponding response of the obtained model to the test input is shown at Figure 6. The mean-root-square error of this model is of 11.5%.

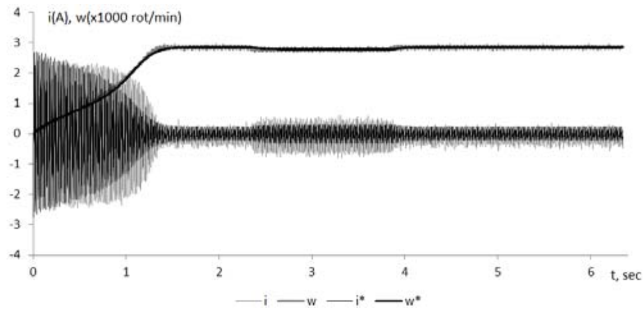

Fig.6. Output variables of the object. Measured experimental values are shown in grey; values calculated using model (6) are shown in black.

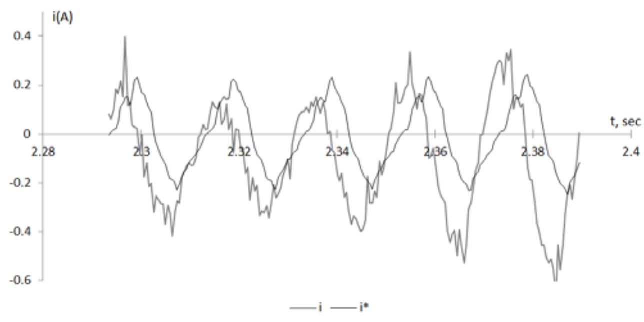

Fig. 7. Enlarged fragment of Pic. 6. at the time of the mechanical load increase

As it can be seen, the model accuracy is significantly improved by adding four terms. It is unexpected to see the term $\beta_0 \left(x_2^{(k)}\right)^3$ because it represents the dependency between the current and the rotor rotation speed in the form $i \sim \omega^3$ which can hardly be explained based on physical phenomena. But as it turned out, such dependency was really present in input data as a result of systematic error caused by the imperfection of the used equipment. This term was removed from the model at the later stages the model construction.

The analysis of this model shows that it does not correctly depict the current amplitude. In particular, it pertains to its increase when mechanical load increases (the middle of the Fig. 6). Additionally, it has some phase shift, which can be seen on the enlarged fragment shown in Figure 7.

To improve the model, one more state variable was added at the third stage. The resulting model form is as follows:

$$\begin{cases} x_1^{(k+1)} = G_{11}u^{(k)} + \alpha_1 x_2^{(k)} u^{(k)} + \\ \quad + \beta_1 \left(x_1^{(k)}\right)^3 + \beta_0 \left(x_2^{(k)}\right)^3 \\ x_2^{(k+1)} = F_{22}x_2^{(k)} + G_{22}M^{(k)} + \alpha_2 x_1^{(k)} u^{(k)} + \\ \quad + \beta_2 \left(x_1^{(k)}\right)^2 x_2^{(k)} + \beta_3 x_1^{(k)} x_2^{(k)} u^{(k)} \\ x_3^{(k+1)} = F_{33}x_3^{(k)} + G_{31}u^{(k)} \\ i^{(k+1)} = x_1^{(k+1)} + C_{13}x_3^{(k+1)} \\ \omega^{(k+1)} = x_2^{(k+1)} \end{cases} \tag{7}$$

The model identification was done starting from the initial approximation of the coefficients taken from the model (6). The mean-root-square error of the resulting model is of 8.3%. The model correctly depicts the phase of the current but still does not depict the increase of its amplitude when the mechanical load increases. Thus at the next stage the evolutionary selection of the optimal model form was conducted. The search was also done within the cubic approximation but that time the state variables vector included 3 components. The model (7) was taken as the initial approximation.

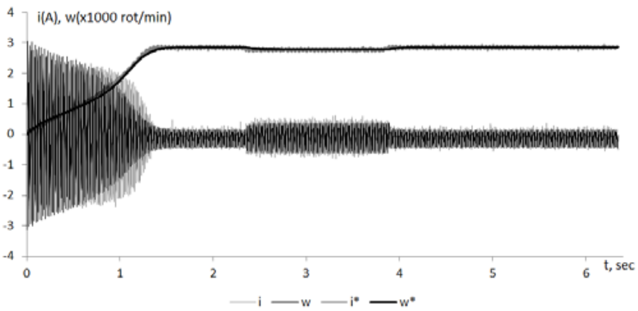

Fig. 8. Output variables of the object. Measured experimental values are shown in grey; values calculated using model (8) are shown in black

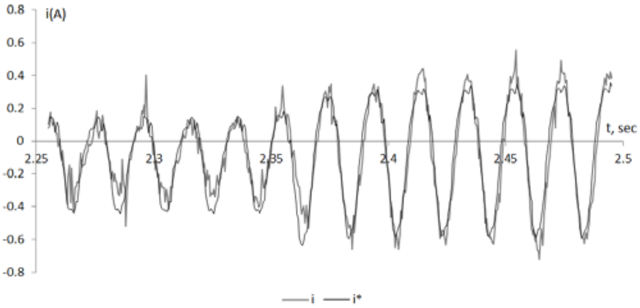

Fig. 9. Enlarged fragment of Figure 8. at the time of the mechanical load increase

This evolutionary model form selection requires more calculations than the one performed at the second stage (due to higher complexity of the model and greater number of alternative model forms). Thus PC cluster containing seven 4-core CPUs was used. The obtained model has the following form:

$$\begin{cases} x_1^{(k+1)} = G_{11}u^{(k)} + \alpha_1 x_2^{(k)} u^{(k)} + \\ \quad + \beta_1 \left(x_1^{(k)}\right)^3 + \beta_0 \left(x_2^{(k)}\right)^3 \\ x_2^{(k+1)} = F_{22}x_2^{(k)} + G_{22}M^{(k)} + \alpha_2 x_1^{(k)} u^{(k)} + \\ \quad + \beta_2 \left(x_1^{(k)}\right)^2 x_2^{(k)} + \beta_3 x_1^{(k)} x_2^{(k)} u^{(k)} \\ x_3^{(k+1)} = F_{33}x_3^{(k)} + G_{31}u^{(k)} + \alpha_3 x_2^{(k)} u^{(k)} + \\ \quad + \beta_4 \left(x_1^{(k)}\right)^2 u^{(k)} + \beta_5 x_2^{(k)} u^{(k)} M^{(k)} \\ i^{(k+1)} = x_1^{(k+1)} + C_{13}x_3^{(k+1)} \\ \omega^{(k+1)} = x_2^{(k+1)} \end{cases} \tag{8}$$

The response of the obtained model to the test input is shown at Figure 8 and 9. The mean-root-square error of this model is of 6.3%.

At the final stage the term $\beta_0 \left(x_2^{(k)}\right)^3$ was excluded from the model and the coefficients refinement was performed. Thus the final model has the following form:

$$\begin{cases} x_1^{(k+1)} = G_{11}u^{(k)} + \alpha_1 x_2^{(k)} u^{(k)} + \beta_1 \left(x_1^{(k)}\right)^3 \\ x_2^{(k+1)} = F_{22}x_2^{(k)} + G_{22}M^{(k)} + \alpha_2 x_1^{(k)} u^{(k)} + \\ \quad + \beta_2 \left(x_1^{(k)}\right)^2 x_2^{(k)} + \beta_3 x_1^{(k)} x_2^{(k)} u^{(k)} \\ x_3^{(k+1)} = F_{33}x_3^{(k)} + G_{31}u^{(k)} + \alpha_3 x_2^{(k)} u^{(k)} + \\ \quad + \beta_4 \left(x_1^{(k)}\right)^2 u^{(k)} + \beta_5 x_2^{(k)} u^{(k)} M^{(k)} \\ i^{(k+1)} = x_1^{(k+1)} + C_{13}x_3^{(k+1)} \\ \omega^{(k+1)} = x_2^{(k+1)} \end{cases} \tag{9}$$

The model coefficients are as follows:

(10)

$$\begin{array}{lll} G_{11} = 0.0123 & \alpha_1 = -4.01 \cdot 10^{-6} & \beta_1 = -0.0368 \\ G_{22} = -0.373 & \alpha_2 = -0.00619 & \beta_2 = -0.00185 \\ G_{31} = 0.00071 & \alpha_3 = -1.93 \cdot 10^{-7} & \beta_3 = 1.53 \cdot 10^{-5} \\ F_{22} = 0.99836 & & \beta_4 = 4.78 \cdot 10^{-5} \\ C_{13} = -0.859 & & \beta_5 = 5.1 \cdot 10^{-8} \end{array}$$

Conclusion

An optimization approach for mathematical model construction with the described additions, in particular an expert analysis, evolutionary selection of an optimal model form and parallelization of calculations allows us to solve the practical task of building models for quite complicated nonlinear dynamical systems. It is obvious that the obtained model can be further improved. But each model is a compromise between simulation precision and model complexity.

The model has been tested under different conditions, in particular with lower input voltage and with higher mechanical load. In both cases the simulation results match the real object behaviour.

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