Marek LANDOWSKI

Maritime University of Szczecin

doi:10.15199/48.2017.01.16

RDM interval method for solving quadratic interval equation

Abstract. *The main task of uncertainty theory is to find the solution with uncertain variable. The ways of uncertainty description are probability density distribution, possibility distribution or interval. To solve the problem with uncertainty variable the calculation on interval is needed. The article presents the usage of RDM interval arithmetic for solving quadratic interval equation. The results obtained from examples are compared with Moore's standard interval arithmetic solutions.*

Streszczenie. *Głównym zadaniem teorii niepewności jest znalezienie rozwiazania ze zmienną niepewną. Niepewność można zapisać w postaci rozkładu gęstości prawdopodobieństwa, rozkładu możliwości lub przedziału. Do rozwiązania zadania ze zmienną niepewną potrzebne są obliczenia na przedziałach. Artykuł przedstawia wykorzystanie arytmetyki interwałowej RDM do rozwiązania interwałowych równań kwadratowych. Wyniki otrzymane z przykładów porównano z rozwiązaniami standardowej arytmetyki interwałowej Moore'a.* ***(Metoda interwałowa RDM do rozwiązywania interwałowych równań kwadratowych)***

Keywords: standard interval arithmetic, RDM interval arithmetic, quadratic interval equation, uncertainty theory
Słowa kluczowe: standardowa artytmetyka interwałowa, arytmetyka interwałowa RDM, interwałowe równanie kwadratowe, teoria niepewności

Introduction

Interval arithmetic is a very important part in uncertainty theory where variable are often not exactly known. Interval arithmetic is used in Granular Computing [9], fuzzy arithmetic [3], automatic control theory [13], probabilistic arithmetic [14], Grey Systems [6]. Interval arithmetic is used where calculation on uncertain variable is made. Quadratic interval equations are met in practice when some of coefficients are uncertain. The problem is to find the region with the roots of polynomial [1, 2].

Moore interval arithmetic [7, 8] is commonly used in practice, but gives a good solution only in specific situations for specific problems.

The alternative for Moore interval arithemtic is RDM interval arithmetic [10, 11, 12, 4, 5]. RDM means Relative Distance Measure. RDM interval arithemtic describes a given interval in the form of set (1).

(1) $$X = [\underline{x}, \overline{x}] = \{x : x = \underline{x} + \alpha_x(\overline{x} - \underline{x}), \alpha_x \in [0, 1]\}$$

The basic operations and their properties in RDM interval arithmetic are described in [4].

For intervals $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$ and the base operations $* \in \{+, -, \cdot, /\}$ span is an interval defined as (2), operation / is defined only if $0 \notin Y$.

(2) $$s(X * Y) = [\min\{X * Y\}, \max\{X * Y\}]$$

The solution obtained by RDM interval arithmetic can be presented in the form of a fomula, illustration, span, cardinality distribution or the center of gravity.

The solution of quadratic interval equation by RDM interval arithmetic

The quadratic interval equation is given by (3):

(3) $$[\underline{a}, \overline{a}]\, x^2 + [\underline{b}, \overline{b}]\, x = [\underline{c}, \overline{c}]$$

Intervals $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$ and $C = [\underline{c}, \overline{c}]$ are presented in RDM notation (4).

(4) $$\begin{aligned} A &= [\underline{a}, \overline{a}] = \{a : a = \underline{a} + \alpha_a(\overline{a} - \underline{a}), \alpha_a \in [0, 1]\} \\ B &= [\underline{b}, \overline{b}] = \{b : b = \underline{b} + \alpha_b(\overline{b} - \underline{b}), \alpha_b \in [0, 1]\} \\ C &= [\underline{c}, \overline{c}] = \{c : c = \underline{c} + \alpha_c(\overline{c} - \underline{c}), \alpha_c \in [0, 1]\} \end{aligned}$$

For every $a \in A$, $b \in B$ and $c \in C$ equation (3) in RDM notation has a form of (5).

(5)
$$[\underline{a} + \alpha_a(\overline{a} - \underline{a})]\, x^2 + [\underline{b} + \alpha_b(\overline{b} - \underline{b})]\, x = \underline{c} + \alpha_c(\overline{c} - \underline{c})$$

where $\alpha_a \in [0, 1]$, $\alpha_b \in [0, 1]$, $\alpha_c \in [0, 1]$, $0 \notin A$.

To find solution of equation (5) first we should find a value of $\Delta = B^2 - 4AC$. For interval third degree multinomial Δ in RDM notation takes the form of (6).

(6)
$$\Delta = \left[\underline{b} + \alpha_b\left(\overline{b} - \underline{b}\right)\right]^2 - 4\left[\underline{a} + \alpha_a(\overline{a} - \underline{a})\right]\left[\underline{c} + \alpha_c(\overline{c} - \underline{c})\right]$$

For $\Delta > 0$ there are solutions in the form (7).

(7) $$x_1 = \frac{\left[-\underline{b} - \alpha_b\left(\overline{b} - \underline{b}\right)\right] - \sqrt{\Delta}}{2[\underline{a} + \alpha_a(\overline{a} - \underline{a})]},\ x_2 = \frac{\left[-\underline{b} - \alpha_b\left(\overline{b} - \underline{b}\right)\right] + \sqrt{\Delta}}{2[\underline{a} + \alpha_a(\overline{a} - \underline{a})]}$$

where $\alpha_a \in [0, 1]$, $\alpha_b \in [0, 1]$, $\alpha_c \in [0, 1]$, $0 \notin A$.

For $\Delta = 0$ the solution has a form of (8).

(8) $$x = \frac{-\underline{b} - \alpha_b\left(\overline{b} - \underline{b}\right)}{2\left[\underline{a} + \alpha_a(\overline{a} - \underline{a})\right]}$$

where $\alpha_a \in [0, 1]$, $\alpha_b \in [0, 1]$, $\alpha_c \in [0, 1]$, $0 \notin A$.

If $\Delta < 0$ there is no real solution.

To find minimum and maximum value of the solution of equation (3) we should consider border value of RDM variables α_a, α_b and α_c, Table 1.

Table 1. Values of variables a, b, c and x for border values of RDM-variables α_a, α_b and α_c.

α_a	0	0	0	1
α_b	0	0	1	0
α_c	0	1	0	0
a	$\underline{a}$	$\underline{a}$	$\underline{a}$	$\overline{a}$
b	$\underline{b}$	$\underline{b}$	$\overline{b}$	$\underline{b}$
c	$\underline{c}$	$\overline{c}$	$\underline{c}$	$\underline{c}$
Δ	$\underline{b}^2 - 4\underline{a}\underline{c}$	$\underline{b}^2 - 4\underline{a}\overline{c}$	$\overline{b}^2 - 4\underline{a}\underline{c}$	$\underline{b}^2 - 4\overline{a}\underline{c}$
If $\Delta > 0$				
x_1	$\frac{-\underline{b} - \sqrt{\Delta}}{2\underline{a}}$	$\frac{-\underline{b} - \sqrt{\Delta}}{2\underline{a}}$	$\frac{-\overline{b} - \sqrt{\Delta}}{2\underline{a}}$	$\frac{-\underline{b} - \sqrt{\Delta}}{2\overline{a}}$
x_2	$\frac{-\underline{b} + \sqrt{\Delta}}{2\underline{a}}$	$\frac{-\underline{b} + \sqrt{\Delta}}{2\underline{a}}$	$\frac{-\overline{b} + \sqrt{\Delta}}{2\underline{a}}$	$\frac{-\underline{b} + \sqrt{\Delta}}{2\overline{a}}$
If $\Delta = 0$				
x	$-\underline{b}/2\underline{a}$	$-\underline{b}/2\underline{a}$	$-\overline{b}/2\underline{a}$	$-\underline{b}/2\overline{a}$
α_a	0	1	1	1
α_b	1	0	1	1
α_c	1	1	0	1
a	$\underline{a}$	$\overline{a}$	$\overline{a}$	$\overline{a}$
b	$\overline{b}$	$\underline{b}$	$\overline{b}$	$\overline{b}$
c	$\overline{c}$	$\overline{c}$	$\underline{c}$	$\overline{c}$
Δ	$\overline{b}^2 - 4\underline{a}\overline{c}$	$\underline{b}^2 - 4\overline{a}\overline{c}$	$\overline{b}^2 - 4\overline{a}\underline{c}$	$\overline{b}^2 - 4\overline{a}\overline{c}$
If $\Delta > 0$				
x_1	$\frac{-\overline{b} - \sqrt{\Delta}}{2\underline{a}}$	$\frac{-\underline{b} - \sqrt{\Delta}}{2\overline{a}}$	$\frac{-\overline{b} - \sqrt{\Delta}}{2\overline{a}}$	$\frac{-\overline{b} - \sqrt{\Delta}}{2\overline{a}}$
x_2	$\frac{-\overline{b} + \sqrt{\Delta}}{2\underline{a}}$	$\frac{-\underline{b} + \sqrt{\Delta}}{2\overline{a}}$	$\frac{-\overline{b} + \sqrt{\Delta}}{2\overline{a}}$	$\frac{-\overline{b} + \sqrt{\Delta}}{2\overline{a}}$
If $\Delta = 0$				
x	$-\overline{b}/2\underline{a}$	$-\underline{b}/2\overline{a}$	$-\overline{b}/2\overline{a}$	$-\overline{b}/2\overline{a}$

In case $\Delta > 0$ there exist two sets of soluions x_1 and x_2 (formula (7)). The spans of solutions of equation (3) are presented in (9).

$$s(x_1) = [x_{1\min}, x_{1\max}]\,,\ s(x_2) = [x_{2\min}, x_{2\max}] \tag{9}$$

where $x_{1\min}$ and $x_{2\min}$ are minimum, $x_{1\max}$ and $x_{2\max}$ are maximum of elements in a row x_1 and x_2 in Table 1, respectively.

In case $\Delta = 0$ there exists one set of soluions x (formula (8)). The span, maximum and minimum value of solutions, of equation (3) is given by (10).

$$s(x) = [x_{\min}, x_{\max}] \tag{10}$$

where $x_{\min}$ and $x_{\max}$ are minimum and maximum of elements in row x in Table 1.

Examples of quadratric interval equation

In example 1 the quadratic interval equation with dependent coefficients has been solved.

Example 1. An object moves under gravity. The task is to find the distance $x \in X$ for given height $y = 2m$ from the start point $(x, y) = (0, 0)$. Fig. 1 shows trajectory of movement of the object in 2-dimensional space $X \times Y$. The object has a constant acceleration $g = 9.8m/s^2$, velocity in the direction x is $u \in U = [11, 13]m/s$, and in the direction y is $v \in V = [15, 17]m/s$.

The position of the object at time t is shown by formula (11):

$$x = ut,\ \ y = vt - \frac{1}{2}gt^2 \tag{11}$$

Formula (11) in the form of quadratic equation is given by (12)

$$y = \frac{v}{u}x - \frac{g}{2u^2}x^2 \tag{12}$$

For $u \in U = [11, 13]$, $v \in V = [15, 17]$, $g = 9.8$ and $y = 2$ formula (12) has a form of (13)

$$2 = \frac{[15, 17]}{[11, 13]}x - \frac{9.8}{2[11, 13]^2}x^2 \tag{13}$$

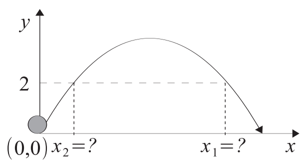

Fig. 1. The trajectory of object movement in 2D space from the starting point $(x, y) = (0, 0)$.

Solution 1 (Moore's interval arithmetic). The quadratic equation obtained from (13) is given by (14)

$$[-0.0405, -0.02899]x^2 + [1.1538, 1.5455]x - 2 = 0 \tag{14}$$

Delta for equation (14) equals (15)

(15)
$$\Delta = [1.1538, 1.5455]^2 - 4[-0.0405, -0.02899](-2) = [1.0074, 2.1565]$$

The solution is calculated from (16)

(16)
$$x_1 = \frac{-[1.1538,1.5455]-\sqrt{\Delta}}{2[-0.0405,-0.02899]},\ x_2 = \frac{-[1.1538,1.5455]+\sqrt{\Delta}}{2[-0.0405,-0.02899]}$$

So the solution of equation (13) obtained by Moore interval arithmetic is given by (17).

$$x_1 \approx [26.64, 51.98],\ \ x_2 \approx [-5.43, 9.34] \tag{17}$$

Solution 2 (RDM interval arithmetic). To find the solution by RDM interval arithmetic first we should write the interval $U = [11, 13]$ and $V = [15, 17]$ in RDM notation, equations (18)

$$\begin{aligned} U &= [11, 13] = \{u : u = 11 + 2\alpha_u, \alpha_u \in [0, 1]\} \\ V &= [15, 17] = \{v : v = 15 + 2\alpha_v, \alpha_v \in [0, 1]\} \end{aligned} \tag{18}$$

For $u \in U$ and $v \in V$ equation (19) represents equation (12) in RDM notation.

$$\frac{-9.8}{2(11 + 2\alpha_u)^2}x^2 + \frac{15 + 2\alpha_v}{11 + 2\alpha_u}x - 2 = 0 \tag{19}$$

where $\alpha_u \in [0, 1]$ and $\alpha_v \in [0, 1]$.

So coefficients of quadratic equation (13) are:

$$a = \frac{-9.8}{2(11 + 2\alpha_u)^2},\ b = \frac{15 + 2\alpha_v}{11 + 2\alpha_u},\ c = -2.$$

From equation (6) Δ is calculated:

$$\Delta = \left(\frac{15 + 2\alpha_v}{11 + 2\alpha_u}\right)^2 - 4\frac{-9.8}{2(11 + 2\alpha_u)^2}(-2)$$

In this case for any $\alpha_u \in [0, 1]$ and $\alpha_v \in [0, 1]$ we have $\Delta > 0$, so solutions of equation (19) are described by (7) in the form of (20).

$$x_1 = \frac{-\frac{15+2\alpha_v}{11+2\alpha_u} - \sqrt{\Delta}}{\frac{-9.8}{(11+2\alpha_u)^2}},\ x_2 = \frac{-\frac{15+2\alpha_v}{11+2\alpha_u} + \sqrt{\Delta}}{\frac{-9.8}{(11+2\alpha_u)^2}} \tag{20}$$

Table 2 presents the solution (20) for border value of RDM variables α_u and α_v.

Table 2. The solution of equation (13) for border values of RDM variables α_u and α_v.

α_u	0	0	1	1
α_v	0	1	0	1
a	-0.0405	-0.0405	-0.02899	-0.02899
b	1.3636	1.5455	1.1538	1.3077
c	-2	-2	-2	-2
Δ	1.5355	2.0645	1.0994	1.4781
x_1	32.1367	36.8220	37.9797	43.5169
x_2	1.5368	1.3413	1.8162	1.5851

From Table 2 (rows x_1 and x_2) the spans of solutions x_1 and x_2 have been found, equation (21).

$$\begin{aligned} s(x_1) &= [x_{1\min}, x_{1\max}] \approx [32.1367, 43.5169] \\ s(x_2) &= [x_{2\min}, x_{2\max}] \approx [1.3413, 1.8162] \end{aligned} \tag{21}$$

Fig. 2 shows surface of solution of equation (13).

Fig. 3 presents distribution of 10 mln random solutions of equation (13) obtained by RDM method. Random solutions were obtained from (20) for random RDM variables $\alpha_u \in [0, 1]$ and $\alpha_v \in [0, 1]$.

Comments to the solution 1 and 2 of example 1. In equation (13) the coefficient a depends on the value b, and the value b depends on the a. Both coefficients include value u, see equation (12).

Solution x_2 obtained with Moore arithmetic has negative values, in this problem $x < 0$ is impossible. In RDM method solutions are greater then 0.

The results obtained with Moore arithmetic are one-dimensional, the RDM arithmetic gives a multidimensional solution.

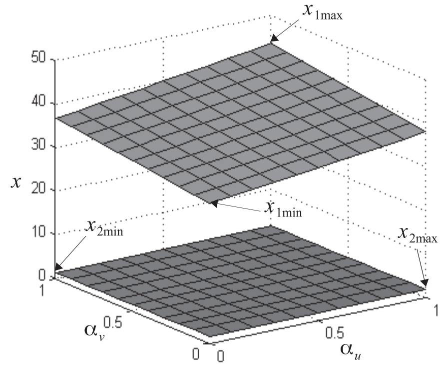

Fig. 2. Illustration of solution of nonlinear interval equation (13) with minimum and maximum values obtained by RDM method.

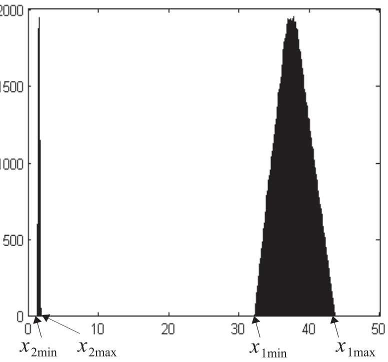

Fig. 3. Distribution of 10 mln random solutions of interval equation (13) obtained by RDM method.

Comparing the spans of the solutions, in the results obtained by Moore arithemtic the span in x_1 and x_2 are larger than in RDM solutions. Moore arithmetic solves equations with dependent coefficients and equations where coefficients a and b are not dependent (value u in a is defferent than u in coefficient b). To explain this situation let us consider the equation (14), here the intervals $[-0.0405, -0.02899]$ and $[1.1538, 1.5455]$ calculated from formula (13) are independent. The dependency between coefficients a and b is not kept in every step in the calculation with Moore arithmetic. The solutions (17) include the results obtained from equations that are not considered in the problem. To prove that the Moore arithmetic solves problems that do not belong to example 1 we can find the solution of (12) where u in the coefficients are different. For constant value $v = 15$, we take in the coefficient a value $u = 11$, but in the coefficient b value $u = 13$, see equation (22).

(22) $$-2 = \frac{15}{13}x - \frac{9,8}{2 \cdot 11^2}x^2$$

The solution of equation (22) is $x_1 \approx 1.85$ and $x_2 \approx 26.64$. That is why the span of solutions obtained by Moore arithmethic is larger than in the RDM arithmetic. The obtained solutions from (22) exist in Moore arithmetic solutions (17), but do not exist in RDM interval arithmetic solutions (21). Moore arithmetic solves equations where coeffiecient a and b are independent, it is a different task than given in example 1. Moore arithmetic cannot solve more complicated problems correctly.

Moreover, in Moore arithmetic, if the coefficients are expressed as intervals, while solving quadratic equation the different value can be taken for calculating Δ and different in the formula for the solution x, see the simple example 2.

Example 2. Find the solution of quadratic interval equation (23)

(23) $$[4, 6]x^2 + [11, 13]x - [3, 5] = 0$$

Solution 1 (Moore's interval arithmetic). Coefficients of quadratic equation $ax^2 + bx + c = 0$ are given by (24)

(24) $$a \in A = [4, 6],\ b \in B = [11, 13],\ c \in C = [-5, -3]$$

Delta has a value of (25)

(25) $$\Delta = [11, 13]^2 - 4[4, 6][-5, -3] = [169, 289]$$

Solutions are calculated from formula (26)

(26) $$x_1 = \frac{-[11, 13] - \sqrt{\Delta}}{2[4, 6]},\ x_2 = \frac{-[11, 13] + \sqrt{\Delta}}{2[4, 6]}$$

The solution obtained by Moore's inerval arithmetic is presented by (27)

(27) $$x_1 = [-3.75, -2],\ x_2 = [0, 0.75]$$

Solution 2 (RDM interval arithmetic). The coefficients of equation (23) in the form of RDM notation are presented in (28)

(28) $$\begin{aligned} A = [4, 6] = \{a : a = 4 + 2\alpha_a, \alpha_a \in [0, 1]\} \\ B = [11, 13] = \{b : b = 11 + 2\alpha_b, \alpha_b \in [0, 1]\} \\ C = [-5, -3] = \{c : c = -5 + 2\alpha_c, \alpha_c \in [0, 1]\} \end{aligned}$$

For $a \in A$, $b \in B$ and $c \in C$ the quadratic equation (23) has a form in RDM notation as given by (29)

(29) $$(4 + 2\alpha_a)x^2 + (11 + 2\alpha_b)x - 5 + 2\alpha_c = 0$$

where $\alpha_a, \alpha_b, \alpha_c \in [0, 1]$.

From formula (6) the value of Δ has a form of (30)

(30) $$\Delta = (11 + 2\alpha_b)^2 - 4(4 + 2\alpha_a)(-5 + 2\alpha_c)$$

where $\alpha_a, \alpha_b, \alpha_c \in [0, 1]$.

The solution obtained by RDM interval arithmetic is calculated from (31)

(31) $$x_1 = \frac{-(11+2\alpha_b)-\sqrt{\Delta}}{2(4+2\alpha_a)},\ x_2 = \frac{-(11+2\alpha_b)+\sqrt{\Delta}}{2(4+2\alpha_a)}$$

where $\alpha_a, \alpha_b, \alpha_c \in [0, 1]$.

The minimum and maximum value of the solution are found by considering the border values of RDM variables α_a, α_b and α_c, Table 3.

Table 3. Values of coefficients a, b, c and solution x for border values of RDM-variables α_a, α_b and α_c.

α_a	0	0	0	1	0	1	1	1
α_b	0	0	1	0	1	0	1	1
α_c	0	1	0	0	1	1	0	1
a	4	4	4	6	4	6	6	6
b	11	11	13	11	13	11	13	13
c	-5	-3	-5	-5	-3	-3	-5	-3
Δ	201	169	249	241	217	193	289	241
x_1	0.3972	0.25	0.3475	0.377	0.2164	0.241	0.3333	0.2103
x_2	-3.1472	-3	-3.5975	-2.2104	-3.4664	-2.0744	-2.5	-2.377

Fig. 4 presents solution of equation (23) obtained by RDM method with maximum and minimum values of solutions.

Fig. 5 presents histogram of distribution obtained from 10 mln random solutions of equation (23) by RDM method. 10 mln random solutions have been obtained from (31) for random variables $\alpha_a, \alpha_b, \alpha_c \in [0, 1]$.

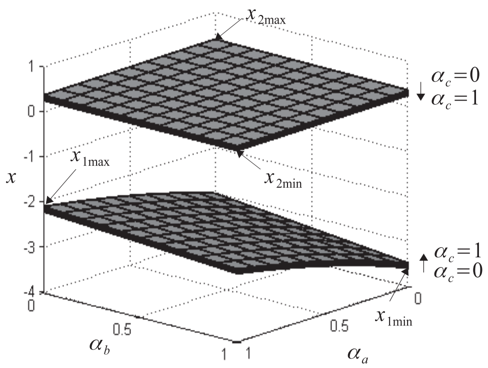

Fig. 4. Illustration of solution of quadratic interval equation (23) obtained by RDM multidimensional arithmetic with maximum and minimum values of solutions.

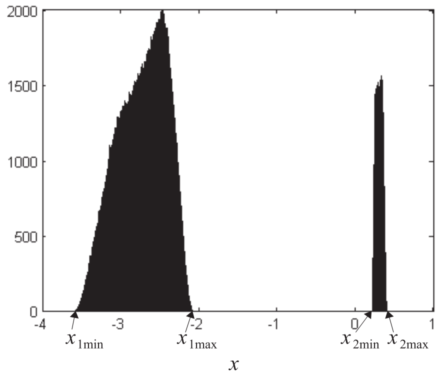

Fig. 5. Distribution of 10 mln random solutions of equation (23) obtained by RDM arithmetic.

The span of the solution obtained by RDM arithmetic presents equation (32)

(32)
$$s(x_1) \approx [-3.5975, -2.0744],\ s(x_2) \approx [0.2103, 0.3972]$$

Comments to the solution 1 and 2 of example 2. In example 2, similarly to solutions of example 1, the solution (27) of Moore arithmetic has a larger span than the solution (32) by RDM arithmetic. The reason of this situation is that in some cases Moore arithmetic during the calculation value of Δ, x_1 and x_2 takes a different value of a coefficient. For example let us consider a special case of equation (23), so let us find the solution of equation (33)

(33)
$$[4, 6]x^2 + 11x - 3 = 0$$

Calculating Δ we take $a = 4$, but for x_1 and x_2 we take $a = 6$. For $a = 4$ value $\Delta = 169$ and the results are (34)

(34)
$$x_1 = -2,\ x_2 = 0.1667$$

Obtained solution (34) is not correct because in the calculation two different values of a have been taken. As we can see the solution (34) is in the solution obtained by Moore's interval arithmetic (27), but solution (34) is not in the solution by RDM interval arithmetic (32). The result of quadratic interval equation obtained by Moore's interval atithmetic is not correct, because it contains values that are not solutions.

Conclusions

The article presents the method for solving quadratic interval equation. The obtained solutions by RDM interval arithmetic are multidimensional, the Moore interval arithmetic solutions are intervals. The given examples show that the Moore's interval arithmetic gives incorrect solutions. In tasks when coefficients are dependent, Moore's arithmetic finds the solution of equation also with independent coefficients, which is incorrect (Example 1). While solving quadratic interval equation the interval coefficient of equation is used more than once, in these situations Moore's arithemtic can take different values for the given coefficient, which is incorrect (Example 2). In every step of the calculation the value of a coefficient should be the same. So the solution obtained by Moore's interval arithmetic contains correct values and values that do not belong to the considered problem. RDM interval method using RDM variable $\alpha \in [0, 1]$ has a possibility to find a full correct solution.

REFERENCES

[1] Ferreira J., Patricio F., Oliveira F.: On the computation of solutions of systems of interval polynomial equations, Journal of Computational and Applied Mathematics, (173), pp. 295-302, 2005.
[2] Ferreira J., Patricio F., Oliveira F.: A priori estimates for the zeros of interval polynomials, Journal of Computational and Applied Mathematics, 136, pp. 271-281, 2001.
[3] Hanss M.: Applied fuzzy arithmetic, Springer-Verlag, Berlin, Heidelberg, 2005.
[4] Landowski M.: Differences between Moore and RDM interval arithmetic, Proceedings of the 7th International Conference Intelligent Systems IEEE IS'2014, September 24-26, 2014, Warsaw, Poland, Volume 1: Mathematical Foundations, Theory, Analyses, Intelligent Systems'2014, Advances in Intelligent Systems and Computing, Vol. 322, pp. 331-340, Springer International Publishing Switzerland, 2015.
[5] Landowski M.: RDM interval arithmetic for solving economic problem with uncertain variables, Logistyka, 6/2014, pp. 13502-13508, 2014.
[6] Liu S., Lin Forest J.Y.: Grey systems, theory and applications, Springer, Berlin, Heidelberg, 2010.
[7] Moore R.E., Kearfott R.B., Cloud J.M.: Introduction to interval analysis, SIAM, Philadelphia, 2009.
[8] Moore R.E.: Interval analysis, Prentice Hall, Englewood Clis, New Jersey, 1966.
[9] Pedrycz W., Skowron A., Kreinovich V. (eds): Handbook of granular computing, Wiley, Chichester, England, 2008.
[10] Piegat A., Landowski M.: Aggregation of inconsistent experts opinions with use of horizontal intuitionistic membership functions, In Atanassov K.T., Castillo O., Kacprzyk J., et al (Eds), Novel Developments in Uncertainty Representation and Processing, Springer, Series Advances in Intelligent Systems and Computing, Vol.401, pp. 215-224, 2016.
[11] Piegat A., Landowski M.: Horizontal membership function and examples of its applications, International Journal of Fuzzy Systems, 17(1), pp. 22-30, March 2015.
[12] Piegat A., Landowski M.: Two interpretations of multidimensional RDM interval arithmetic - multiplication and division, International Journal of Fuzzy Systems, 15(4), pp. 488-496, Dec. 2013.
[13] Sohn C.: On damping ratio of polynomials with perturbed coefficients, IEEE Trans. Autom. Control, 37(2), 1992.
[14] Williamson R.: Probabilistic arithmetic, Ph.D. thesis, Department of Electrical Engineering, University of Queensland, 1969.

Author: *Ph.D. Marek Landowski, Department of Mathematical Methods, Maritime University of Szczecin, Waly Chrobrego 1-2, 70-500 Szczecin, Poland, email: m.landowski@am.szczecin.pl*