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Oscillation Theorems for Third Order Neutral Delay Difference Equation with Negative Coefficient in the Neutral Term

***Abstract**. The article describes the new oscillation criteria which improves the existing results for the third order neutral delay difference equation* $\Delta^2\left(a_n\Delta\left(x_n - p_n x_{n-l}\right)\right) - q_n f(x_{n-m}) = 0$ *with the negative coefficient in the neutral term is obtained. Where* $l, m > 0, \{p_n\} and \{q_n\}$ *are positive sequences.*

Streszczenie.** W artykule opisano nowe kryteria oscylacji, które poprawiają dotychczasowe wyniki dla równania różnicy opóźnienia neutralnego trzeciego rzędu* $\Delta^2\left(a_n\Delta\left(x_n - p_n x_{n-l}\right)\right) - q_n f(x_{n-m}) = 0$ *z ujemnym współczynnikiem w członie neutralnym. Gdzie są sekwencje dodatnie.* ***(Twierdzenia o oscylacji dla równania różnicy neutralnego opóźnienia trzeciego rzędu z ujemnym współczynnikiem w członie neutralnym)

Keywords: Oscillation, Neutral delay, Third – order, Positive Sequence
Słowa kluczowe: oscylacje, neutralne równanie różnicowe

1.Introduction

Here some new sufficient conditions which insure that every solution of the above equation either oscillates or non oscillates is given. [1] R. Grace, John R.Garef discussed about the Oscillatory Behaviour of Third order nonlinear differential equations with a nonlinear non positive neutral term

[2] Osama Moaaz, discusses about the oscillation criteria of third order Neutral Delay Differential Equations. [5] Ozkan Ocalan extensively discusses the problem of Oscillation of neutral differential equation with positive and negative coefficients Here we provided oscillation results in third order difference equation based on the existence of third order differential equations. .Consider the third order Neutral Delay Difference Equation,

(1.1) $\Delta^2\left(a_n\Delta\left(x_n - p_n x_{n-l}\right)\right) - q_n f(x_{n-m}) = 0$

With the negative coefficient in the neutral term, Where $l, m > 0, \{p_n\}$ and $\{q_n\}$ are positive sequences.

Theorem 1.1
Every solution of the equation(1.1) is oscillatory, for some $\rho_n p_n > 0, f_{nm} > f_n f_m, l, m > 0,$ if

$$\frac{2a_{n+1}w_{n+1}q_{n+1}f(z_{n+1-m})}{\rho_{n+1}} < \frac{a_{n+2}w_{n+2}q_{n+2}f(z_{n+2-m})}{\rho_{n+2}}$$

where $z_n = x_n - p_n x_{n-l}, w_n = \frac{\rho_n \Delta z_n}{q_n f(z_{n-m})}$.

Proof: Suppose x_n be a non-oscillatory solution

Without the loss of generality, let us assume that x_n is eventually positive solution of equation (1.1). If $z_n = x_n - p_n x_{n-l}$, then equation (1.1) becomes $\Delta^2\left(a_n\Delta(z_n)\right) - q_n f(z_{n-m} + p_{n-m}x_{n-m-l}) = 0$

(1.2) $\Delta\left(a_{n+1}\Delta z_{n+1} - a_n\Delta z_n\right) - q_n f(z_{n-m} + p_{n-m}x_{n-m-l}) = 0$

$$\Delta\left(a_{n+2}\Delta z_{n+2} - a_{n+1}\Delta z_{n+1} - a_{n+1}\Delta z_{n+1} + a_n\Delta z_n\right) - q_n f(z_{n-m} + p_{n-m}x_{n-m-l}) = 0$$

If $w_n = \frac{\rho_n \Delta z_n}{q_n f(z_{n-m})}$ then $w_n > 0$

Then equation (1.2) becomes,

$$\frac{a_{n+2}w_{n+2}q_{n+2}f(z_{n+2-m})}{\rho_{n+2}} - 2\frac{a_{n+1}w_{n+1}q_{n+1}f(z_{n+1-m})}{\rho_{n+1}} + \frac{a_n w_n q_n f(z_{n-m})}{\rho_n} - q_n f(z_{n-m} + p_{n-m}x_{n-m-l}) = 0$$

$$\frac{a_n w_n q_n f(z_{n-m})}{\rho_n} = q_n f(z_{n-m} - p_{n-m}x_{n-m-l}) - \frac{a_{n+2}w_{n+2}q_{n+2}f(z_{n+2-m})}{\rho_{n+2}} + 2\frac{a_{n+1}w_{n+1}q_{n+1}f(z_{n+1-m})}{\rho_{n+1}}$$

From the condition

$$\frac{2a_{n+1}w_{n+1}q_{n+1}f(z_{n+1-m})}{\rho_{n+1}} < \frac{a_{n+2}w_{n+2}q_{n+2}f(z_{n+2-m})}{\rho_{n+2}}$$

$$\frac{a_n w_n q_n f(z_{n-m})}{\rho_n} < q_n f(z_{n-m} + p_{n-m}x_{n-m-l})$$

$$w_n < \frac{q_n f(z_{n-m} + p_{n-m}x_{n-m-l})\rho_n}{a_n q_n f(z_{n-m})}$$

$$w_n < \frac{q_n f(z_{n-m}) f(1 + p_{n-m}x_{n-m-l})\rho_n}{a_n q_n f(z_{n-m})}$$

$$w_n < \frac{f(1 + p_{n-m}x_{n-m-l})\rho_n}{a_n}$$

$$\frac{f(1 + p_{n-m}x_{n-m-l})\rho_n}{a_n} > w_n$$

Generalizing

$$\sum_{s=n_0}^{n} \frac{\rho_s}{a_s} f(1 + p_{s-m}x_{s-m-l}) > w_s$$

When $s \to \infty$

$$\sum_{s=n_0}^{n} \frac{\rho_s}{a_s} f(1 + p_{s-m}x_{s-m-l}) > w_s$$

Which is not possible
Hence every solution of equation (1.1) is oscillatory.

Theorem 1.2
If x_n is an eventually positive solution of equation (1.1) and

(1.3) $z_n = x_n - p_n x_{n-l}$

then for sufficiently large n, the condition $z_n > 0, \Delta(a_n \Delta z_n) > 0$ exists.

Proof: Let x_n is an eventually positive solution of equation (1.1). Then there exists $n_1 \geq n_0$ such that $x_{n-l} > 0$ for $n \geq n_1$. From the definition of z_n, it is clear that $z_n > 0$, $n \geq n_1$

We claim that $\Delta(a_n \Delta z_n) > 0, n \geq n_2$

Suppose $\Delta(a_n \Delta z_n) \leq 0$ for $n \geq n_2$, Since $a_n > 0$

Then $a_{n+1}\Delta z_{n+1} - a_n \Delta z_n \leq 0$

$a_{n+1}\Delta z_{n+1} \leq a_n \Delta z_n$

$$\Delta z_{n+1} \leq \frac{a_n}{a_{n+1}} \Delta z_n$$

Taking summation from n_2 to n,

$$\sum_{n_2}^{n} \Delta z_{n+1} \leq \sum_{n_2}^{n} \frac{a_n}{a_{n+1}} \Delta z_n$$

$$\sum_{n_2}^{n} \Delta z_{n+1} \leq \frac{a_{n_2}}{a_{n_2+1}} \Delta z_{n_2} + \frac{a_{n_2+1}}{a_{n_2+2}} \Delta z_{n_2+1} + \frac{a_{n_2+2}}{a_{n_2+3}} \Delta z_{n_2+2} + \ldots\ldots + \frac{a_n}{a_{n+1}} \Delta z_n$$

$$\sum_{n_2}^{n} z_{n+2} - z_{n+1} \leq \frac{a_{n_2}}{a_{n_2+1}} \Delta z_{n_2} + \frac{a_{n_2+1}}{a_{n_2+2}} \Delta z_{n_2+1} + \frac{a_{n_2+2}}{a_{n_2+3}} \Delta z_{n_2+2} + \ldots\ldots + \frac{a_n}{a_{n+1}} \Delta z_n$$

$z_n \to -\infty$ as $n \to \infty$

Which contradicts the equation (1.3)

Hence we have $\Delta(a_n \Delta z_n) > 0$ for all n.

Theorem 1.3

If x_n is an eventually negative solution of equation (1.1) and z_n is defined by equation (1.3).then for sufficiently large n, the condition $z_n < 0, \Delta(a_n \Delta z_n) < 0$ exists.

Proof: Let x_n is an eventually negative solution of equation (1.1). Then there exists $n_1 \geq n_0$ such that $x_{n-l} < 0$ for $n \geq n_1$.

From the definition of z_n, it is clear that $z_n < 0$ (1.4)

We claim that $\Delta(a_n \Delta z_n) < 0, n \geq n_2$

Suppose $\Delta(a_n \Delta z_n) \geq 0$ for $n \geq n_2$

Since $a_n > 0$,then $a_{n+1}\Delta z_{n+1} - a_n \Delta z_n \geq 0$

$a_{n+1}\Delta z_{n+1} \geq a_n \Delta z_n$

$$\Delta z_{n+1} \geq \frac{a_n}{a_{n+1}} \Delta z_n$$

Taking summation from n_2 to n

$$\sum_{n_2}^{n} \Delta z_{n+1} \geq \sum_{n_2}^{n} \frac{a_n}{a_{n+1}} \Delta z_n$$

$$\sum_{n_2}^{n} \Delta z_{n+1} \geq \frac{a_{n_2}}{a_{n_2+1}} \Delta z_{n_2} + \frac{a_{n_2+1}}{a_{n_2+2}} \Delta z_{n_2+1} + \frac{a_{n_2+2}}{a_{n_2+3}} \Delta z_{n_2+2} + \ldots\ldots + \frac{a_n}{a_{n+1}} \Delta z_n$$

$$\sum_{n_2}^{n} z_{n+2} - z_{n+1} \geq \frac{a_{n_2}}{a_{n_2+1}} \Delta z_{n_2} + \frac{a_{n_2+1}}{a_{n_2+2}} \Delta z_{n_2+1} + \frac{a_{n_2+2}}{a_{n_2+3}} \Delta z_{n_2+2} + \ldots\ldots + \frac{a_n}{a_{n+1}} \Delta z_n$$

$z_n \to \infty$ as $n \to \infty$

Which contradicts the equation (1.4)

Theorem 1.4

Assume $f(x_{n-m}) = x_{n-m}, p_n = 0, a_n = 1$ equation (1.1) becomes,

(1.5) $\Delta^3 x_n - q_n(x_{n-m}) = 0, n \in N(n_0)$

If x_n is an eventually positive solution of the equation (1.5) then there exist $x_n \geq x_{n-k} > 0, \Delta^2 x_n > 0$, for all $n \geq n_0$

Proof

From the equation (1.5)

$\Delta^3 x_n = q_n(x_{n-m}) > 0, n \in N(n_0)$

$\Delta^2(x_{n+1} - x_n) = q_n(x_{n-m}) > 0, n \in N(n_0)$

Hence

(1.6) $\Delta^2 x_{n+1} > \Delta^2 x_n$

So $\Delta^2 x_n$ is eventually increasing. Since q_n is a positive function, the decreasing function $\Delta^2 x_n$ is either eventually positive or eventually negative.

Suppose there exist $n_2 \geq n_1$ such that $\Delta^2 x_{n_2} < 0.$

Taking summation from n_2 to s the equation (1.6) becomes,

$$\sum_{n=n_2}^{s} \Delta^2 x_{n+1} > \sum_{n=n_2}^{s} \Delta^2 x_n$$

$$\sum_{n=n_2}^{s} \Delta(x_{n+2} - x_{n+1}) > \sum_{n=n_2}^{s} \Delta(x_{n+1} - x_n)$$

$$\sum_{n=n_2}^{s} x_{n+3} - x_{n+2} - \Delta x_{n+1} > \sum_{n=n_2}^{s} x_{n+2} - x_{n+1} - x_{n+1} + x_n$$

$$-x_{n_2+2} + x_{n_2+1} + x_{n_2+4} - x_{n_2+3} + \ldots\ldots\ldots + x_{s+3} - x_{s+2} - x_{n+2} + x_{s+1} > \sum_{n=n_2}^{s} x_{n+2} - x_{n+1} - y_{xn+1} + x_n$$

$$> -x_{n+1} + \ldots\ldots + x_{s+2} - x_{s+1} - (-x_{n_2} + x_{n_2+2} + \ldots\ldots\ldots + x_{s+1} - x_n)$$

$$> -x_{n_2+1} + x_{s+2} - x_{s+1} + x_{n_2}$$

$$-x_{n_2+2} + x_{s+3} + x_{n_2+1} - x_{s+2} > -x_{n_2+1} + x_{s+2} - x_{s+1} + x_{n_2}$$

$$x_{s+3} - x_{s+2} - x_{s+2} + x_{s+1} > x_{n_2+2} - x_{n_2+1} - x_{n_2+1} + x_{n_2}$$

$$x_{s+3} - x_{s+2} - \Delta x_{s+1} > \Delta x_{n_2+1} - \Delta x_{n_2}$$

$$x_{s+3} > \Delta x_{n_2+1} - \Delta x_{n_2} + x_{s+2} + \Delta x_{s+1}$$

When $n \to \infty$ and $x_n > \infty$

Which is a contradiction. Hence $\Delta^2 x_n > 0$, for all $n \geq n_0$.

Conclusions:

In this paper some new oscillation criteria for third order neutral delay difference equation is obtained by utilising summation average techniques and comparison principle. The aim of this study is to develop the new criteria for the oscillation, so that we apply them when other criteria fails. In future, we extend this results for higher order neutral delay difference equations.

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