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A review and analysis of Quade's fundamental geometric time domain concept for the summation of non-active powers of poly-phase systems

Abstract. *How to determine the total non-active power of arbitrary periodical poly-phase loads or in other words: how to sum non-active powers resulting from non-sinusoidal and unbalanced voltages and currents? With algebraic summation or via aggregate power like the standards propose? In the time domain or with harmonic decomposition? What is the genuine meaning of non-active and apparent power? The reader may be amazed by questioning these problems which seem to be solved. Instead this article shows that the general solution is not that of the standards which define limiting cases but one that exists since a long time in the form of the geometric power concept of W. Quade that is commonly unkown today. The geometric method is compared to the concepts of aggregate power (Rechtleistung) and the algebraic summation of fictitious non-active powers. The consequences and meaning of the different concepts are analyzed.*

Streszczenioe. *Jak wyznaczyć całkowitą moc nieczynną dowolnych okresowych odbiorników wielofazowych, czyli inaczej: jak zsumować moce nieczynne wynikające z niesinusoidalnych i niezrównoważonych napięć i prądów? Z sumowaniem algebraicznym czy za pomocą sumarycznej mocy, jak proponują normy? W dziedzinie czasu czy z rozkładem harmonicznym? Jakie jest prawdziwe znaczenie nieaktywnej i pozornej mocy? Czytelnik może być zdumiony kwestionowaniem tych problemów, które wydają się być rozwiązane. Zamiast tego artykuł ten pokazuje, że generalnym rozwiązaniem nie jest rozwiązanie norm definiujących przypadki graniczne, ale takie, które istnieje od dawna w postaci koncepcji geometrycznej potęgi W. Quade, która jest dziś powszechnie nieznana. Metodę geometryczną porównuje się z pojęciami zagregowanej mocy (Rechtleistung) i algebraicznym sumowaniem fikcyjnych mocy nieczynnych. Analizowane są konsekwencje i znaczenie różnych pojęć.* (***Koncepcja w dziedzinie czasu do sumowania mocy nieczynnych układów wielofazowych***)

Keywords: power, vector space, orthogonality, interference, geometric summation, poly-phase circuit, fictitious starpoint
Słowa kluczowe: Sumowanie mocy, przestrzeń wektorowa, obwody wielofazowe

Introduction

The determination (measurement) of the total non-active power magnitude Q_{Σ} of a poly-phase system requires the consideration of superposition (interference) of non-active power oscillations $Q_{\mu k}(t)$ of the linearly independent circuit loops. This is because non-active power is not an average value (not a scalar quantity) like active power. The algebraic summation of non-active power magnitudes is generally wrong. It is shown that the interferences are considered by the geometric method established by *W. Quade*. The total powers summed geometrically will be called "actual" powers in the following because they determine the condition of the load in terms of power. We distinguish the "actual" powers from technical power definitions (like aggregate power) related to optimal utilization and compensation concepts that aim to "condition" the load seen by the supply grid by means of component systems (goal: symmetry and sinusoidality). We "unmask" the technical apparent and non-active power definitions as limiting cases. The geometric power concept presented explains the basic nature of non-active power and provides the poly-phase generalization of the single-phase power orthogonality relation, i.e. it provides the summation law of non-active powers.

Two-pole power orthogonality relation and vector space – *Quade's* geometric concept

It is commonly agreed that in the special case of single-phase systems (two-pole, one linearly independent loop), figure 1, the power orthogonality relation for the root mean squares (RMS) of periodical voltage and current oscillations U and I holds:

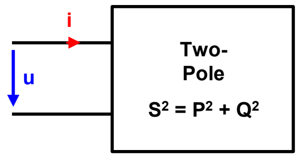

Fig. 1. Electric two-pole (single-phase system) with periodical voltage and current

(1) $$U^2 \cdot I^2 = S^2 = P^2 + Q^2$$

S is the apparent power, P the active power and Q the non-active power (the term "reactive" power shall be used only in the sinusoidal case). *S. Fryze* showed that the load current oscillation of the two-pole can always be split into two orthogonal components [1], [2] and therefore proved the general validity of (1). One current component (i_{prop}) is proportional, the other (i_{orth}) is orthogonal to the periodic supply voltage oscillation. Active (transformable) power results from the identity

(2) $$P = \frac{1}{T}\int_0^T u \cdot i \; dt = \frac{1}{T}\int_0^T u \cdot i_{prop} \; dt = U \cdot I_{prop}$$

For non-active power no such identity exists because of the orthogonality relation

(3) $$0 = \frac{1}{T}\int_0^T u \cdot i_{orth} \; dt$$

There is no useful energy transformation (no work done) over one or more (N) full periods N·T. The value Q is determined solely by the RMS of the orthogonal current component: $Q = U \cdot I_{orth}$. It indicates that part of the load current uselessly loads the supply, causes losses and reduces utilization. The RMS of proportional and orthogonal current components are related by $I^2 = I_{prop}^2 + I_{orth}^2$ which leads to equation (1) when multiplied with the square of the RMS two-pole voltage. Of course the power orthogonality relation does not provide information about how active and non-active powers are constituted (phase shift, harmonic content).

The non-active power value Q is a measure for the degree of non-proportionality of voltage and current oscillations of a circuit (linearly independent loop).

The instantaneous power p can be split accordingly into proportional and orthogonal power components:

(4) $$p = u \cdot i = u \cdot i_{prop} + u \cdot i_{orth} = p_{prop}(t) + p_{orth}(t)$$

The first summand of p generates a positive power oscillation $p_{prop}(t)$ around it's average, the active power P. Only equal order harmonics of voltage and current are contained. The second summand is a power oscillation $p_{orth}(t)$ with average zero, so with no useful effect. Harmonics of equal and unequal order are contained. p(t) has got negative parts if $p_{orth}(t)$ is greater than zero instantaneously; in other words: the negative parts of p represent non-active (non-transformable) power. Figure 2 illustrates the power oscillations for a stationary sinusoidal condition (index 1). Note that Q_1 is the peak value of $p_{orth}(t)$ that appears instantaneously, not as an average.

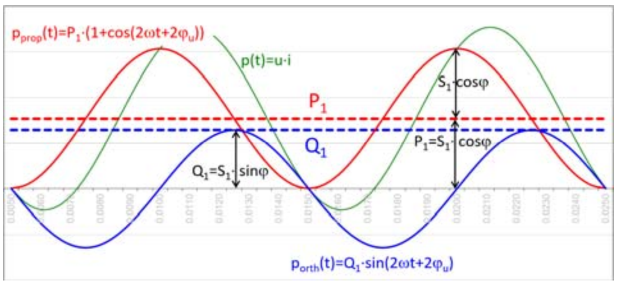

Fig. 2. Power oscillations, sinusoidal case

The power oscillations $p_{prop}(t)$ and $p_{orth}(t)$ do also appear at each linearly independent loop of a multi-pole. Due to the connection of the linearly independent loops of poly-phase systems the non-active power oscillations in loops superimpose (interfere with) each other. This central property must be considered in a basic power theory. It is insufficient to regard magnitudes from RMS only when non-active powers are summed.

W. Quade showed that the power orthogonality relation that results from orthogonal functions corresponds to a formulation in vector space [3 – 5]. It is shown in the following how the interference of oscillations is considered by the vector space Ansatz. Unfortunately this has not been widely recognized to date and it is therefore the aim of this article to bring *Quade's* method to attention (which was published only in German 1933-1937). The presentation provided here can not be found in *Quade's* original publications.

The correspondence is based on the properties of a (*Euclidean*) vector space of all real-valued continuous functions *f(t)*, *g(t)* on a closed interval [a,b] which is an inner product space, where an inner (scalar) product is defined by

(5) $$\vec{f} \cdot \vec{g} = \left\langle \vec{f}, \vec{g} \right\rangle := \int_a^b f(t) \cdot g(t)\; dt$$

Orthogonality is defined by this inner product: in a *Euclidean* vector space the vectors $\vec{f}, \vec{g}$ are orthogonal if $\vec{f} \perp \vec{g} \Leftrightarrow \vec{f} \cdot \vec{g} = 0$. The inequality of *Cauchy-Schwarz* holds:

$\vec{f}^2 \cdot \vec{g}^2 - \vec{f} \cdot \vec{g} = \begin{vmatrix} \vec{f}^2 & \vec{f} \cdot \vec{g} \\ \vec{g} \cdot \vec{f} & \vec{g}^2 \end{vmatrix} \geq 0$, where the determinant is called *Gram*'s determinant. This is only equal to zero if the vectors are linearly dependent, i.e. proportional. *Gram*'s determinant can be extended to a n-dimensional vector space and thus generalizes the inequality of *Cauchy-Schwarz*. The connection of orthogonal functions, vector space and power is obvious and allows to apply the laws of vector algebra to the determination of powers as follows. A three-dimensional *Euclidean* vector space is sufficient. The scalar product of the spatial vectors (not phasors) of voltage $\vec{U}$ and current $\vec{I}$ equals the temporal average of the product of their time functions u and i, this is the active power P of the period T:

(6) $$P = \vec{U} \cdot \vec{I} = \frac{1}{T} \int_0^T u \cdot i\; dt$$

Active power is zero if $\vec{U} \perp \vec{I}$, then $\vec{U} \cdot \vec{I} = 0$, and accordingly if the functions u and i are orthogonal. The magnitude of a voltage or current vector $\vec{V}$ equals the RMS of it's time function v(t), this is V, of the period T:

(7) $$V = \left|\vec{V}\right| = \sqrt{\vec{V} \cdot \vec{V}} = \sqrt{\frac{1}{T} \int_0^T v^2 dt}$$

Apparent power is the product of the magnitudes of voltage and current vectors:

(8) $$S = \left|\vec{U}\right| \cdot \left|\vec{I}\right|$$

It is obviously equal to the scalar product of the collinear voltage and current vectors. The inequality of *Cauchy-Schwarz* applies accordingly:

(9) $$\left|\vec{U}\right| \cdot \left|\vec{I}\right| \geq \left|\vec{U} \cdot \vec{I}\right|$$

$P = S$ or $I_{prop} = I$ happens only if the vectors resp. oscillations of voltage and current are proportional, i.e. having equal curve form and being in phase, a proportionality of the oscillations on an instantaneous basis. This condition only occurs at an ohmic resistor.

Consequently, as is well known, the meaning of apparent power S at the two-pole is that it equals the active power of the equivalent proportional circuit ($S=P_{prop}$) which represents the limiting case of complete energy transformation in the load (if the load could be made proportional) with $\lambda=P/S=1$, depicted in the following figure 3.

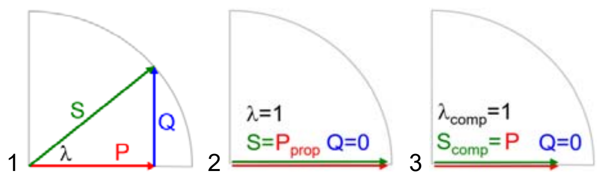

Fig. 3. Power orthogonality relation: 1 general, 2 equiv. proportional case, 3 compensated case

In practice a load will have a certain steady proportion of active and non-active power, say $S_{Load\text{-}initial}$. It can not be made proportional. If the non-active power is compensated at the terminals then the grid currents supplied to the load reduce to the "active" currents. Thus the supply is disburdened from the non-active load currents (and related losses) but the (internal) condition of the load does not change. Only the grid sees that now $S_{Load\text{-}comp} = P_{Load}$ and $S_{Load\text{-}comp} < S_{Load\text{-}initial}$. The compensated load is the new equivalent proportional system as seen by the supply grid.
Now it is essential to notice that in three-dimensional inner product spaces the specialized form of the identity of *Lagrange* holds:

(10) $$\left|\vec{U}\right|^2 \cdot \left|\vec{I}\right|^2 = \left(\vec{U} \cdot \vec{I}\right)^2 + \left|\vec{U} \times \vec{I}\right|^2$$

This is the power orthogonality relation in vector space. By comparing relation (10) with relation (1) we find

(11a) $$\left|\vec{U} \times \vec{I}\right| \equiv Q$$

This is the central conclusion. In vector space formalism non-active power is determined by the vector product (cross product) of the vectors of voltage and current:

(11b) $$\vec{Q} = \vec{U} \times \vec{I} = -\vec{I} \times \vec{U}$$

The scalar product $\vec{U} \cdot \vec{I} = U \cdot I \cdot \cos(\vec{U}, \vec{I})$ is the generalization of the sinusoidal $P_1 = U_1\ I_1\ \cos(\varphi_1)$ and the vector product $\vec{U} \times \vec{I} = U \cdot I \cdot \sin(\vec{U}, \vec{I})$ is the generalization of the sinusoidal $Q_1 = U_1\ I_1\ \sin(\varphi_1)$.
The vector character of Q is not just a formality as will become clear in the following and it leads to the following visualization as depicted in figure 4.

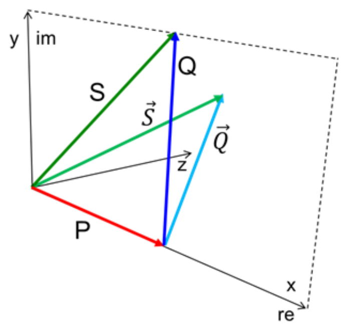

Fig. 4. Visualization of the vector character of non-active power

In the sinusoidal regime Q and S lie in the complex plane and Q is always parallel to the imaginary axis (im, y). With harmonics present the vectors $\vec{Q}$ and $\vec{S}$ extend into the third dimension (z). P always lies on the real axis (re, x) and is orthogonal to Q and to $\vec{Q}$. $\vec{Q}$ always lies in a plane parallel to the y-z-plane.

Total "actual" power of poly-phase systems (multi-poles)

In this context a multi-pole (n–pole), figure 5, is a AC poly-phase system (e.g. polygon or star connected or a combination) with imposed periodical supply voltages of 360/n degrees phase shift and the condition that the instantaneous sum of all pole ("phase" or "line") currents and of all phase-phase voltages is zero (a neutral conductor is not a voltage "phase"). k is the number of the voltage measurement reference conductor (which can be anyone of the poles). A power per phase is not measurable without further provisions if there is no physical neutral conductor N. For the sake of simplicity but without lack in generality the following derivations are basically that of the three-pole.

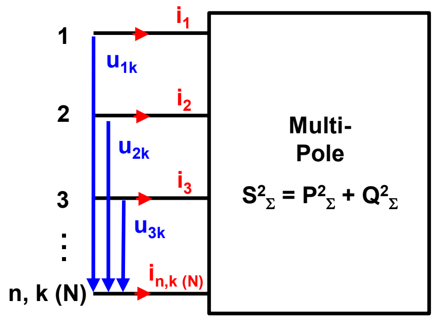

Fig. 5. Electric n-pole in general

The total instantaneous power of an arbitrary circuit with m impedances Z_ν is the sum of the instantaneous powers of the individual impedances of the circuit, simultaneously measured:

(12a) $$p_\Sigma = \sum_{\nu=1}^{m} u_\nu \cdot i_\nu$$

Active and non-active power at each individual impedance can be determined and then summed. But this is not practicable. It is also not necessary because it is equivalent to determine the total active, non-active and apparent power with a measurement of the voltages and currents at the terminals as indicated in figure 5.
With n terminals n–1 linearly independent loop voltages and loop currents can be measured. The total instantaneous power of a multi-pole with n terminals and arbitrary internal structure is completely determined by the sum of the n–1 linearly independent loop instantaneous powers related to an arbitrary reference conductor k:

(12b) $$p_\Sigma = \sum_{\mu=1}^{n} u_{\mu k} \cdot i_\mu$$

where $u_{\mu k} \cdot i_\mu = 0$ if μ = k. If there is a neutral conductor N then the reference conductor is N. A three-phase system without neutral conductor has got 2 linearly independent loops, e.g. 1-2 and 3-2, the one with neutral conductor has got 3 linearly independent loops: 1-N, 2-N, 3-N. The instantaneous „*Aron*" power of the three-pole for conductor number k as the reference and μ ≠ k is:

(12c) $p_\Sigma = u_{21} \cdot i_2 + u_{31} \cdot i_3$, k=1 or

(12d) $p_\Sigma = u_{12} \cdot i_1 + u_{32} \cdot i_3$, k=2 or

(12e) $p_\Sigma = u_{13} \cdot i_1 + u_{23} \cdot i_2$, k=3

Note that $u_{12} = -u_{21}$ etc. The instantaneous total power of the four-pole with neutral conductor k:=N as the reference and μ ≠ k is:

(12f) $$p_\Sigma = u_{1N} \cdot i_1 + u_{2N} \cdot i_2 + u_{3N} \cdot i_3$$

Energy is a scalar quantity and the total active power is the algebraic sum of the active powers of the linearly independent loops:

(13) $$P_\Sigma = \sum_{\mu=1}^{n} \frac{1}{T} \int_0^T p_{\mu k}(t)dt = \sum_{\mu=1}^{n} \frac{1}{T} \int_0^T p_{prop-\mu k}(t)dt = \sum_{\mu=1}^{n} P_{\mu k}$$

For the three-pole: $P_\Sigma = P_{21} + P_{31} = P_{12} + P_{32} = P_{13} + P_{23}$.

How to sum the non-active powers of linearly independent circuit loops? The integration of the $p_{orth\text{-}\mu k}(t)$ results in zero and is a "dead end". The following was *Quade's* assessment of the situation [5] (author's translation from German):

"*Given an arbitrary AC network with non-sinusoidal voltages and currents: according to which law do non-active powers and apparent powers sum? The question is easily answered for the active powers because due to the conservation of energy the resulting total active power is the algebraic sum of the single active powers. Because according laws for non-active and apparent powers were not available, all efforts to make progress in the question of definition of non-active and apparent power have failed so far. This gap became especially sensible with the attempt to define the different powers for poly-phase systems.*"

The problem is solved without harmonic decomposition by applying identity (11a) resp. (11b) with which we can sum the vector representations of the $p_{orth\text{-}\mu k}(t)$:

$$\vec{Q}_\Sigma = \sum_{\mu=1}^{n} \vec{Q}_{\mu k} \tag{14}$$

For the three-pole: $\vec{Q}_\Sigma = \vec{Q}_{21} + \vec{Q}_{31} = \vec{Q}_{12} + \vec{Q}_{32} = \vec{Q}_{13} + \vec{Q}_{23}$.

The important point is that the phase information of the $p_{orth\text{-}\mu k}(t)$ is still contained in the vector representation as will become clear in the following.

Now relation (10) is used. The square of the unknown vector $\vec{Q}_\Sigma$ is:

$$Q_\Sigma^2 = \vec{Q}_\Sigma^2 = \left(\vec{Q}_{12} + \vec{Q}_{32}\right)^2 = Q_{12}^2 + Q_{32}^2 + 2 \cdot \vec{Q}_{12} \cdot \vec{Q}_{32} \tag{15}$$

Note that the same procedure is valid for a three-phase system with neutral conductor N:

$$\begin{aligned} Q_\Sigma^2 &= \vec{Q}_\Sigma^2 = \left(\vec{Q}_{1N} + \vec{Q}_{2N} + \vec{Q}_{3N}\right)^2 \\ &= Q_{1N}^2 + Q_{2N}^2 + Q_{3N}^2 + 2 \cdot \vec{Q}_{1N} \cdot \vec{Q}_{2N} \\ &\quad + 2 \cdot \vec{Q}_{1N} \cdot \vec{Q}_{3N} + 2 \cdot \vec{Q}_{2N} \cdot \vec{Q}_{3N} \end{aligned} \tag{16}$$

Scalar products appear in the equation which we may call interference terms that represent the superposition of the oscillations of linearly independent loops. The scalar product is computed by $2 \cdot \vec{Q}_{12} \cdot \vec{Q}_{32} = 2 \cdot Q_{12} \cdot Q_{32} \cdot \cos\delta$ where $\cos\delta$ is the cosine of the angle between both vectors. This phase shift is missing in the simple application of algebraic summation. $\cos\delta$ is zero for 90°, i.e. when both vectors are perpendicular (functions orthogonal). This is the case if only oscillations with 90° phase shift are contained in both $Q_{\mu k}$ or – which is similar – if only harmonics of unequal order are contained. Only then $Q_\Sigma^2 = Q_{12}^2 + Q_{32}^2 + 0$ and each $Q_{\mu k}$ equals an apparent power $S_{\mu k}$. Generally $Q_\Sigma^2 \le \sum_\mu Q_{\mu k}^2$ and thus also $S_\Sigma^2 \le \sum_\mu S_{\mu k}^2$ holds. Also harmonic non-active powers generally sum geometrically because of phase shifts of equal order harmonics of voltage and current of one loop. However, harmonic apparent powers always sum quadratically because they are constituted of the RMS of harmonic voltages U_h and currents I_h and $U_H^2 = \sum_h U_h^2$ and $I_H^2 = \sum_h I_h^2$ holds. The flaw in the algebraic non-active power summation is that Q is not an average like P that covers a time span, e.g. an oscillation period $T=t_2-t_1>0$, but the peak value of an oscillation $p_{orth\text{-}\mu k}(t)$ that appears at just a moment in time $t_2-t_1=0$. The point is that peaks of different oscillations typically appear at different points in time because of phase shifts. Interestingly, it is the shortcoming of *C. Budeanu's* concept [8] (thorougly analysed in [9]) not to consider this and we can easily construct scenarios where the algebraic summation of non-active power magnitudes leads to contradictions.

To get a concrete solution from (15) or (16), the scalar product $\vec{Q}_{12} \cdot \vec{Q}_{32}$ is now expressed by substituting the $\vec{Q}_{\mu k}$ with the vector products of loop voltage and loop current vectors:

$$\begin{aligned} \vec{Q}_{12} \cdot \vec{Q}_{32} &= \left(\vec{U}_{12} \times \vec{I}_1\right) \cdot \left(\vec{U}_{32} \times \vec{I}_3\right) \\ &= \left(\vec{I}_1 \times \vec{U}_{12}\right) \cdot \left(\vec{I}_3 \times \vec{U}_{32}\right) \end{aligned} \tag{17}$$

This is resolved using the identity of *Lagrange* to substitute the unknown vector products by solvable scalar products of measured voltages and currents:

$$\begin{aligned} \left(\vec{U}_{12} \times \vec{I}_1\right) \cdot \left(\vec{U}_{32} \times \vec{I}_3\right) &= \begin{vmatrix} \vec{U}_{12} \cdot \vec{U}_{32} & \vec{U}_{12} \cdot \vec{I}_3 \\ \vec{I}_1 \cdot \vec{U}_{32} & \vec{I}_1 \cdot \vec{I}_3 \end{vmatrix} \\ &= \left(\vec{U}_{12} \cdot \vec{U}_{32}\right) \cdot \left(\vec{I}_1 \cdot \vec{I}_3\right) \\ &\quad - \left(\vec{U}_{12} \cdot \vec{I}_3\right) \cdot \left(\vec{I}_1 \cdot \vec{U}_{32}\right) \end{aligned} \tag{18}$$

If $\vec{U}_{12} = \vec{U}_{32}$ and $\vec{I}_1 = \vec{I}_3$ then the specialized form of the identity of *Lagrange* follows and thus the two-pole power orthogonality relation. With the non-active power magnitudes squared of the two linearly independent loops $Q_{12}^2 = U_{12}^2 \cdot I_1^2 - P_{12}^2$ and $Q_{32}^2 = U_{32}^2 \cdot I_3^2 - P_{32}^2$ eventually the equation for the actual total non-active power of the three-pole is

$$\begin{aligned} Q_\Sigma^2 &= U_{12}^2 \cdot I_1^2 - P_{12}^2 + U_{32}^2 \cdot I_3^2 - P_{32}^2 \\ &\quad + 2 \cdot \left(\vec{U}_{12} \cdot \vec{U}_{32}\right) \cdot \left(\vec{I}_1 \cdot \vec{I}_3\right) \\ &\quad - 2 \cdot \left(\vec{U}_{12} \cdot \vec{I}_3\right) \cdot \left(\vec{I}_1 \cdot \vec{U}_{32}\right) \end{aligned} \tag{19}$$

By applying the correspondence (5) vector space ⇔ periodical functions, (19) is mapped into

$$\begin{aligned} Q_\Sigma^2 &= U_{12}^2 \cdot I_1^2 - P_{12}^2 + U_{32}^2 \cdot I_3^2 - P_{32}^2 \\ &\quad + 2 \cdot \frac{1}{T}\int_0^T u_{12} \cdot u_{32}\, dt \cdot \frac{1}{T}\int_0^T i_1 \cdot i_3\, dt \\ &\quad - 2 \cdot \frac{1}{T}\int_0^T u_{12} \cdot i_3\, dt \cdot \frac{1}{T}\int_0^T i_1 \cdot u_{32}\, dt \end{aligned} \tag{20}$$

where the integrals are the "interference" terms (the phase information). Equation (20) is valid for any curve form of voltage and current oscillations and is directly measurable without harmonic decomposition. Under sinusoidal conditions the three-dimensional vector space is reduced to two dimensions and becomes equivalent to the complex plane. Then $Q_\Sigma \equiv \mathrm{Im}\left(\underline{U}_{12} \cdot \underline{I}_1^*\right) + \mathrm{Im}\left(\underline{U}_{32} \cdot \underline{I}_3^*\right)$.

If the three-phase system contains a neutral conductor N and k:=N then the result is

$$\begin{aligned} Q_\Sigma^2 &= U_{1N}^2 \cdot I_1^2 - P_{1N}^2 + U_{2N}^2 \cdot I_2^2 - P_{2N}^2 \\ &\quad + U_{3N}^2 \cdot I_3^2 - P_{3N}^2 \\ &\quad + 2 \cdot \frac{1}{T}\int_0^T u_{1N} \cdot u_{2N}\, dt \cdot \frac{1}{T}\int_0^T i_1 \cdot i_2\, dt \end{aligned}$$

$$
\begin{aligned}
&- 2 \cdot \frac{1}{T}\int_0^T u_{1N} \cdot i_2 \ dt \cdot \frac{1}{T}\int_0^T i_1 \cdot u_{2N} \ dt \\
(21) \qquad &+ 2 \cdot \frac{1}{T}\int_0^T u_{1N} \cdot u_{3N} \ dt \cdot \frac{1}{T}\int_0^T i_1 \cdot i_3 \ dt \\
&- 2 \cdot \frac{1}{T}\int_0^T u_{1N} \cdot i_3 \ dt \cdot \frac{1}{T}\int_0^T i_1 \cdot u_{3N} \ dt \\
&+ 2 \cdot \frac{1}{T}\int_0^T u_{2N} \cdot u_{3N} \ dt \cdot \frac{1}{T}\int_0^T i_2 \cdot i_3 \ dt \\
&- 2 \cdot \frac{1}{T}\int_0^T u_{2N} \cdot i_3 \ dt \cdot \frac{1}{T}\int_0^T i_2 \cdot u_{3N} \ dt
\end{aligned}
$$

Having derived Q_Σ^2, it is easy to state the expression for S_Σ^2 . With $P_\Sigma^2 = (P_{12} + P_{32})^2 = P_{12}^2 + P_{32}^2 + 2 \cdot P_{12} \cdot P_{32}$ and $S_\Sigma^2 = P_\Sigma^2 + Q_\Sigma^2$ it follows for the three-pole:

$$
\begin{aligned}
S_\Sigma^2 &= U_{12}^2 \cdot I_1^2 + U_{32}^2 \cdot I_3^2 + 2 \cdot P_{12} \cdot P_{32} \\
(22) \qquad &+ 2 \cdot (\vec{U}_{12} \cdot \vec{U}_{32}) \cdot (\vec{I}_1 \cdot \vec{I}_3) \\
&- 2 \cdot (\vec{U}_{12} \cdot \vec{I}_3) \cdot (\vec{I}_1 \cdot \vec{U}_{32})
\end{aligned}
$$

and

$$
\begin{aligned}
S_\Sigma^2 &= U_{12}^2 \cdot I_1^2 + U_{32}^2 \cdot I_3^2 + 2 \cdot P_{12} \cdot P_{32} \\
(23) \qquad &+ 2 \cdot \frac{1}{T}\int_0^T u_{12} \cdot u_{32} \ dt \cdot \frac{1}{T}\int_0^T i_1 \cdot i_3 \ dt \\
&- 2 \cdot \frac{1}{T}\int_0^T u_{12} \cdot i_3 \ dt \cdot \frac{1}{T}\int_0^T i_1 \cdot u_{32} \ dt
\end{aligned}
$$

Eventually we arrive at the multi-pole generalization of the two-pole power orthogonality relation that satisfactorily unifies the description of single- and poly-phases systems:

$$
\begin{aligned}
S_\Sigma^2 &= \left(\sum_{\mu=1}^{n} \vec{U}_{\mu k} \cdot \vec{I}_\mu\right)^2 + \left(\sum_{\mu=1}^{n} \vec{U}_{\mu k} \times \vec{I}_\mu\right)^2 \\
(24) \qquad &= \left(\sum_{\mu=1}^{n} P_{\mu k}\right)^2 + \left(\sum_{\mu=1}^{n} \vec{Q}_{\mu k}\right)^2 = P_\Sigma^2 + \vec{Q}_\Sigma^2 \\
&= P_\Sigma^2 + Q_\Sigma^2
\end{aligned}
$$

with $\mu \neq k$. This is equivalent to the algebraic sum of the active powers and the geometric sum of the non-active powers of the individual impedances of a load. It is the generalization of the complex arithmetic of the sinusoidal case and always yields the actual total powers related to the load impedances for any condition. Remarkably (24), when resolved, contains RMS and temporal averages which can be easily measured with digital meters. The equation continuously reduces into the correct two-pole power orthogonality relation under single-phase conditions (just one loop current not zero) as well as into the sinusoidal result in absence of harmonics.

The geometric summation of non-active powers results in a total apparent multi-pole power S_Σ that equals the total active power of the equivalent proportional poly-phase system (if the load could be made proportional). It represents the limiting case of complete energy transformation in the load, exactly like at the two-pole, and is therefore the closest possible connection to active power (to physics), in other words: it is the "rock bottom" definition of apparent power.

What is the value of the total non-active power for a proportional (ohmic) and asymmetric load? It is zero. This is only natural. The energy transformation is complete in a proportional and asymmetric poly-phase load because $S_\Sigma = P_\Sigma = \sum_\mu I_\mu^2 \cdot R_{Load-\mu}$, thus $Q_\Sigma = 0$ at the terminals. The currents are proportional to the voltages over the load resistances. The indicator for the completeness of energy transformation in the load is $\lambda_\Sigma = P_\Sigma / S_\Sigma \le 1$. There is no "asymmetry non-active power" on physical grounds. The characterization of poly-phase system asymmetry by a non-active power value is only a technical definition. If a poly-phase circuit contains switches (semiconductors) then non-active power will be "generated" by the switching which produces non-proportionality by discontinuity. If the load impedances are ohmic then only active power is transformed there though non-active power from switching is measured at the terminals of the multi-pole. This shows the degree of non-proportionality in the load and thus the "under-utilization".

The rated (layout, setpoint) total apparent power of electrical equipment is defined by the rated values of equipment voltage U_{LLr} and current I_{Lr} for symmetric and sinusoidal conditions ($U_{12r}=U_{23r}=U_{31r}$, $I_{1r}=I_{2r}=I_{3r}$):

$$
(25) \qquad S_{r\Sigma} = \sqrt{3} \cdot U_{LLr} \cdot I_{Lr}
$$

Contrary to the two-pole where the apparent power represents the equipment load as well, the total apparent power S_Σ represents the equipment load only in the proportional or in the sinusoidal and symmetric case because of interference. Generally the loop non-active power vectors, or equivalently the non-active power vectors of the individual impedances, are not collinear. Also S_Σ does not indicate an asymmetric condition and how the load is distributed. Therefore it is necessary to assess the apparent power of each linearly independent loop. Each one needs to keep the limit on average, at the three-pole without neutral:

$$
(26) \qquad U_{12} \cdot I_1 \le S_{r\Sigma} / \sqrt{3} \ \text{ and } \ U_{32} \cdot I_3 \le S_{r\Sigma} / \sqrt{3}
$$

At the three-pole with neutral:

$$
(26a) \qquad U_{1N} \cdot I_1 \le S_{r\Sigma} / 3 \ \text{ and } \ U_{2N} \cdot I_2 \le S_{r\Sigma} / 3 \ \text{ and } \ U_{3N} \cdot I_3 \le S_{r\Sigma} / 3
$$

Note that an overload seems to be a nominal load if e.g. $U_{12} \le U_{LLr}$ and $I_1 \ge I_{Lr}$ at the same time and vice versa. The apparent power is equivocal in this case. The powers always need to be measured on the supply side to include the equipment impedance (e.g. on the supply side of a transformer). Overload means that the rated apparent power is exceeded in the temporal average with the consequence that the rated lifetime of the equipment is reduced. How much it is reduced depends on the built-in safety margin (over-dimensioning). Load is determined by temperature rise during onload and offload times (ageing of insulating material), by current (local current density, magnitude of magnetic forces) and by voltage (local electrical field strength, strain of insulating material).
If we want to include the supply system in the evaluation of utilization then we need to distinguish between the transformation of energy in the load and the utilization of energy in the system supply+load. Asymmetric load currents cause losses $\sum_\mu I_\mu^2 \cdot R_{Supply-\mu}$ in the symmetric

energy supply impedances that are greater than with symmetric currents (similarly this applies to the resistances (losses) in the load). Thus the total active power supplied is smaller for an asymmetric load than it could be with the symmetric one (the total instantaneous power is an oscillation with an asymmetric load and can be constant for a symmetric one). The total utilization of the system is not optimal. Therefore the goal of compensation is simply to make the feeding currents proportional to and as symmetric as the symmetric and sinusoidal supply voltages, more exactly: fictitious starpoint voltages (and to stabilize these). How to achieve this result is a technical problem which is not related to the determination of actual total powers of a poly-phase load. A possible solution is provided by the standard DIN 40110-2:2002-11. Before we explore the meaning of the definitions of the standard and the related aggregate power (Rechtleistung) concept, we turn to a commonly used method applied with power measurements in poly-phase systems (but not with compensation concepts) and analyse it's peculiarities.

Algebraic summation concept

The method is the algebraic summation of fictitious phase powers. This requires the provision of a fictitious starpoint "0" as the reference. The phase values are determined with the fictitious starpoint voltages $u_{\mu 0}$ which are computed from the terminal voltages $u_{\mu\nu}$ (including phases and neutral), no physical resistor starpoint is required [7]:

(27) $$u_{\mu 0} = \frac{1}{n} \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^{n} u_{\mu\nu}$$, μ=1..n

Like the phase-phase voltages $u_{\mu\nu}$, the $u_{\mu 0}$ are so called "zero sum quantities" because always $\sum_{\mu=1}^{n} u_{\mu 0} = 0$. Often the voltages $u_{\mu 0}$ are measured against a grounded PT starpoint (e.g. high voltage supply system). This approximates the fictitious starpoint voltages. However the "real" fictitious starpoint voltages (27) could be easily determined by a digital power analyzer. This would simplify the measurement because no special provision needs to be taken for a V-connected PT for example. In a scenario with neutral conductor N the conductor number μ runs from 1 to n and conductor n is the neutral N. Power analyzers support a voltage measurement against a real neutral (for low voltage systems). If this is used then in the following equations the index "μ0" is to be exchanged with "μN" and the conductor number runs from 1 to n-1 only, with N as the reference conductor. Then the following quantities are not fictitious any more.

The phase fictitious active powers are

(28) $$P_{\mu 0} = \frac{1}{T} \int_{0}^{T} u_{\mu 0} \cdot i_{\mu} dt$$

The phase fictitious apparent powers are

(29) $$S_{\mu 0} = U_{\mu 0} \cdot I_{\mu}$$

The phase fictitious non-active powers are

(30) $$Q_{\mu 0} = +\sqrt{S_{\mu 0}^{2} - P_{\mu 0}^{2}}$$

Note that the $Q_{\mu 0}$ are positive magnitudes. The total active power is

(31) $$P_{\Sigma} = \sum_{\mu=1}^{n} P_{\mu 0}$$

and the total fictitious non-active power is defined by

(32) $$Q_{\Sigma 0} = \sum_{\mu=1}^{n} Q_{\mu 0}$$

The total fictitious apparent power results from the two-pole orthogonality relation:

(33) $$S_{\Sigma 0}^{2} = P_{\Sigma}^{2} + Q_{\Sigma 0}^{2}$$

Only if all $S_{\mu 0}$ vectors have equal direction, i.e. all power factors $\lambda_{\mu 0} = P_{\mu 0} / S_{\mu 0}$ are equal, we may calculate $S_{\Sigma 0} = \sum_{\mu=1}^{n} S_{\mu 0}$. What are the basic assumptions of this method? Linearly dependent fictitious starpoint voltages and pole (line) currents are used. Because the fictitious starpoint voltages are always as symmetric as the usually symmetric supply voltages $u_{\mu\nu}$, the asymmetry measured depends basically only on the currents. The phase powers determined do not tell anything about the distribution of power in the load. This would require the knowledge of the voltages and currents of all load impedances. The algebraic summation of the phase fictitious active powers is always correct.

The main premise of this method is the wrong assumption that fictitious non-active powers can as well be summed algebraically in general.

The assumption is only partly correct in the sinusoidal regime because phase shifts are neglected. Their consideration would require the knowledge of the fundamental oscillations of voltages and currents (sign of Q). Under asymmetric condition the summation of magnitudes results in so called "fictitious asymmetry non-active power" (term coined by *M. Depenbrock*) because the fictitious phase to starpoint voltages are proportional to the related pole (and neutral) currents only for an ohmic and symmetric multi-pole or for a compensated one. If the load is non-linear (non-proportional) then the harmonic contents of the currents result in another artificial increase of total non-active power. In poly-phase systems harmonics usually interfere with each other by superposition of oscillations due to the connection of star or polygon because $\sum_{\mu=1}^{n} i_{\mu} = 0$. This effect is neglected by summing magnitudes. The algebraic summation method is frequently used but the described effects are typically not considered. Concluding, from a physical viewpoint this method is correctly usable only for (quite) symmetric and (quite) sinusoidal loads in regards total non-active and apparent powers.

The fictitious starpoint concept, that is part of DIN 40110-2:2002-11, is as well the basis to determine the required (shunt) compensator currents. In this context the standard also includes the definition of "aggregate" or "collective" voltage, current and apparent power (originally: "Rechtleistung", literally "right power" [5, 10], term coined by *F. Emde* and first used by *W. Quade*). The fictitious starpoint and aggregate power concept has been compared formally to the geometric concept with mathematical rigour by *H.D. Fischer* [6]. The aggregate power concept is examined in the following and it's meaning is analysed.

“Aggregate” power concept

This is a technical concept. Aggregate power results from quadratic averaging of linearly dependent RMS values of the fictitious starpoint voltages and pole currents. Aggregate non-active power is a derived quantity, not one resulting from first principles like in *W. Quade’s* geometric concept.

The aggregate voltage of a three-pole is

(34a)
$$U_{\Sigma Agg} = \sqrt{U_{10}^2 + U_{20}^2 + U_{30}^2} = \sqrt{1/3\left(U_{12}^2 + U_{23}^2 + U_{31}^2\right)}$$

For a four-pole with a neutral conductor N it is

(34b)
$$U_{\Sigma Agg} = \sqrt{U_{10}^2 + U_{20}^2 + U_{30}^2 + U_{N0}^2} = \sqrt{1/4\left(U_{12}^2 + U_{23}^2 + U_{31}^2 + U_{1N}^2 + U_{2N}^2 + U_{3N}^2\right)}$$

The aggregate current of a three-pole is

(35a)
$$I_{\Sigma Agg} = \sqrt{I_1^2 + I_2^2 + I_3^2}$$

For a four-pole with a neutral conductor N it is

(35b)
$$I_{\Sigma Agg} = \sqrt{I_1^2 + I_2^2 + I_3^2 + I_N^2}$$

The aggregate power (“Rechtleistung”) is

(36)
$$S_{\Sigma Agg} = U_{\Sigma Agg} \cdot I_{\Sigma Agg}$$

The aggregate non-active power is declared by

(37)
$$Q_{\Sigma Agg}^2 = S_{\Sigma Agg}^2 - P_\Sigma^2$$

P_Σ results from (31). What are the premises and consequences of aggregate power? How is (37) justified? Similar to the algebraic summation, (mainly) current harmonics increase the result of (37) and the quadratic averaging of magnitudes leads to a further increase by asymmetry which is as well “fictitious asymmetry non-active power”. What is the value of $S_{\Sigma Agg}$ if one pole current continuously reduces to zero, i.e. the three-pole reduces into a two-pole? This is a case happening regularly at electric arc furnace operation for example. From a physical (circuit theoretical) or measurement point of view we should expect that then the total apparent power becomes identic to that of the two-pole, like (24) does. But this is not the case for $S_{\Sigma Agg}$. Instead it’s value remains too large by a factor of $\sqrt{2}$ if we assume symmetric phase-phase voltages u_{12}, u_{23}, u_{31}:

(38)
$$S_{\Sigma Agg-1ph} = \sqrt{3 \cdot U_{10}^2} \cdot \sqrt{2 \cdot I^2} = \sqrt{3} \cdot \sqrt{2} \cdot U_{10} \cdot I$$

The reason is the artificial quadratic averaging obviously. $S_{\Sigma Agg-1ph}$ does not equal the apparent power of the two-pole. The single-phase condition illustrates that the aggregate power is not a generally applicable apparent power value in regards load analysis. Instead $S_{\Sigma Agg}$ is a limiting case of S_Σ and $S_{\Sigma Agg} = S_\Sigma$ only if the load is symmetric and sinusoidal. This means that $Q_{\Sigma Agg}$ has got a proper meaning only in the symmetric and sinusoidal case which is indicated by the factors $1/\sqrt{3}$ and $1/2$ in equations (34a) and (34b) that are only meaningful for symmetry. The orthogonality relation (37) is thus justified by the limting case.

$S_{\Sigma Agg}$ equals the total active power of the equivalent symmetric and sinusoidal and proportional poly-phase system [5] and is a special limiting case of S_Σ in which equation (37) is valid. $S_{\Sigma Agg}$ then reduces into the simple form $S_{\Sigma Agg} = 3 \cdot U_{10} \cdot I_1 = \sqrt{3} \cdot U_{12} \cdot I_1$.

Why was aggregate power included in DIN40110-2? Because it is not meant to determine the present (actual) state of a dynamical load, it results from a concept that is rooted in optimal utilization which is achieved when $Q_{\Sigma Agg} = 0$. Optimal utilization occurs if the load is perfectly compensated. The intended (technical) meaning (this is the premise) of aggregate power becomes clear by analysing the definitions of DIN40110-2:2002-11 which are based on the FBD (*Fryze-Buchholz-Depenbrock*) method [11]. Generally speaking, the standard defines how to determine per phase the fictitious active and non-active current components with the fictitious starpoint voltages. These fictitious current components are generally completely different from the genuine proportional and orthogonal components of the load currents which determine the active and non-active powers of the load. The load current components could be determined would the voltage over the load impedances be known. But this is not the purpose of the standard. The purpose is to determine fictitious non-active current components such that the grid currents become symmetric and proportional to the fictitious starpoint voltages that represent the supply voltages. Thus the (shunt) compensation ideally symmetrizes the asymmetric load, even if this is in a single phase condition like with an electric arc furnace. If this is economically meaningful in practice is another question. For compensation the powers are not of interest but current decomposition is. The aggregate non-active power $Q_{\Sigma Agg}$ is determined by the aggregate fictitious non-active current component and the aggregate voltage. It is connected to the aggregate apparent power $S_{\Sigma Agg}$ via the power orthogonality relation. Vice versa, $Q_{\Sigma Agg}$ is computed directly via (37). $S_{\Sigma Agg}$ is just a means to an end to determine $Q_{\Sigma Agg}$ that is basically the required compensation power.

Unreflectedly, a typical user with a typical power analyzer interpretes the measured aggregate power as the actual total apparent power of the load, contrary to the premise on the one hand and contrary to the real meaning as a special limiting case on the other hand. It is a shortcoming of DIN40110-2:2002-11 not to explain the meaning of aggregate power but just to define it. Also part 2 of the standard is mistaken to be the generalization of part 1 which it is clearly not. This causes confusion. Part 1 presents *Fryze’s* concept for single-phase systems. *Quade’s* concept is the natural generalization to poly-phase systems.

However, in practice usually the aggregate as well as the algebraic power concept are a sufficient approximation to the actual total power if the multi-pole load is (quite) symmetric and (quite) sinusoidal.

Comparing the three "power determination methods" presented, generally $Q_{\Sigma} \le Q_{\Sigma 0} \le Q_{\Sigma Agg}$ holds, thus $S_{\Sigma} \le S_{\Sigma 0} \le S_{\Sigma Agg}$.

Under sinusoidal (with phase shift) and symmetric condition $Q_{\Sigma} = Q_{\Sigma 0} = Q_{\Sigma Agg} \ge 0$.

Under sinusoidal (with phase shift) and asymmetric condition $(Q_{\Sigma} \le Q_{\Sigma 0} \le Q_{\Sigma Agg}) \ge 0$.

Under sinusoidal and proportional (no phase shift) and symmetric condition $Q_{\Sigma} = Q_{\Sigma 0} = Q_{\Sigma Agg} = 0$.

Under proportional and asymmetric condition $Q_{\Sigma} = 0$ and $(Q_{\Sigma 0} < Q_{\Sigma Agg}) > 0$.

An indication for the utilization of energy in the system supply+load are the power factors of each linearly independent loop $\lambda_{\mu k} = P_{\mu k} / (U_{\mu k} \cdot I_{\mu}) \le 1$ which include distortion and represent asymmetry. Only if all $\lambda_{\mu k}$ are unity and all loop voltages and currents are sinusoidal then the utilization of load and supply is optimal. A penalty for harmonic contents or asymmetry can be based on the assessment of the $\lambda_{\mu k}$.

Conclusion

The geometric concept of *W. Quade* is fundamental (built on first principles), illuminating and instructive. It considers the interference (superposition) of oscillations in connected linearly independent circuit loops and determines the actual total non-active and apparent powers of a poly-phase load in arbitrary condition and provides the generalization of the two-pole power orthogonality relation. Generally the total apparent power is equal to the total active power of the equivalent proportional system. Non-active power is a measure for non-proportionality. Logically equation (24) follows. The aggregate power concept of *F. Buchholz* and of DIN 40110-2 is based on the premise of optimal utilization of the power supply by means of compensation. It defines aggregate non-active and apparent power as according secondary indicators that are limiting cases of the actual powers. Apparent and non-active power values of the three concepts described become equal for a sinusoidal (with or without phase shift) and symmetric load. A close physical relation of total non-active and apparent power to active power is only provided by the geometric concept. The aggregate power concept is a technical one (a means to an end). It is a central aim of this article to show this basic difference between the concepts of *Fryze* / *Quade* (physical basis) and *Buchholz* / DIN40110-2 (component systems). The physical connection of the geometric concept has got the consequence that a load needs to be analysed in a multi-dimensional way and not just by one non-active power value. But this is quite natural when analysing complex systems. *W. Quade's* geometric power concept could be easily implemented in digital power analyzers. This would complement the usual measurement of fictitious phase powers and aggregate powers and provide a more comprehensive and theoretically more satisfactory analysis of any poly-phase load.

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