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Forward acoustic problem analysed by Boundary Element Method

***Abstract**. In this paper, the influence of the numerical integration of the singular integrands on the precision of calculation for the forward acoustic problems was presented. The acoustic problem was described by the Helmholtz equation in 2D space. As a state variable, the velocity potential was selected. Analysis was conducted in the frequency domain. Singular integrands were calculated by subtracting out the singular term, which could be calculated analytically. It was proved that such an approach provides high-precision results formulated for acoustic problems, even for the coarse discretization of the boundary. For calculations, the benchmark proposed in [13] was used.*

***Streszczenie.** W tym artykule zbadano wpływ całkowania osobliwego na dokładność obliczeń zagadnienia prostego dla problemów akustycznych. Opis matematyczny sprowadzono do równania Helmholtza w przestrzeni 2D. Jako zmienną stanu wybrano potencjał prędkości przeprowadzając obliczenia w przestrzeni częstotliwości. Osobliwe wyrażenia podcałkowe obliczono metodą odjęcia członu osobliwego tak aby funkcja podcałkowa była nieosobliwa, zaś człon osobliwy, aby było można scałkować analitycznie. Wykazano, że takie podejście zapewnia bardzo dobrą dokładność dla zagadnień prostych formułowanych w akustyce nawet przy bardzo skromnej dyskretyzacji brzegu analizowanego obszaru. Do obliczeń wykorzystano benchmark zaproponowany w pracy [13] (**Analiza zagadnień akustycznych Metodą Elementów Brzegowych**).*

Keywords: Numerical integration of singular integrands, precision of calculation for Helmholtz equation in 2D space, the forward problem for Acoustic Transmission Tomography.
Słowa kluczowe: Całkowanie całek osobliwych, precyzja obliczeń równania Helmholtza w przestrzeni 2D, zagadnienie proste w Transmisyjnej Tomografii Akustycznej.

Introduction

Four different computational methods engage in acoustic analysis and solutions of the inverse problems: ray tracing, FEM, BEM, and DG-FEM (Discontinuous Galerkin Finite-Element Method). In this paper, we would like to focus readers' attention on the BEM formulated for the frequency domain. Formally the mathematical description would be almost identical to Diffuse Optical Tomography, but the physical meaning of the state functions and material coefficient and their units are different [1-12].

Ultrasound tomography models are different from the mathematical models formulated for X-ray tomography models [13-15]. One can distinguish the following modes of Ultrasound Computed Tomography (USCT):

1. transmission mode,
2. reflection mode,
3. diffraction mode.

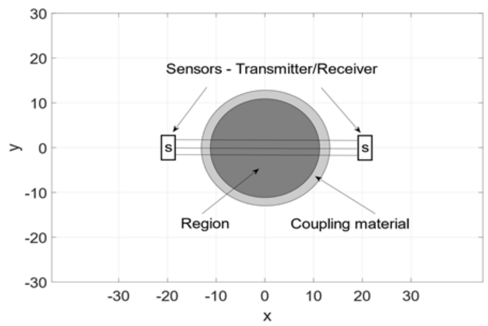

Fig. 1. Sketch of the transmission mode of USCT

In this paragraph, it will be interested only in transmission mode. Using this mode, we can image the speed of sound or the attenuation coefficient of the investigated structure. In the transmission mode of USCT, we have two ways. Compare the amplitudes of the pulses to reconstruct the attenuation coefficient or measure time-of-flight (TOF) to image the speed of the sound. In the following figure, the simplified set used in the transmission mode is shown in Fig. 1. In clinical practice is used the coupling gel to reduce the acoustic impedance [13]. The thin layer of coupling gel is visible in Fig. 1 marked by a grey colour.

Governing equations for the forward acoustic problem

The acoustic field is assumed to be present in the domain of a homogeneous isotropic fluid, and it is modelled by the linear wave equation [13]:

(2) $$\nabla^2\psi(\mathbf{p},t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{p},t)$$

where $\psi(\mathbf{p},t)$ is the scalar time-dependent velocity potential related to the time-dependent velocity vector $\mathbf{v}(\mathbf{p},t)=\nabla\psi(\mathbf{p},t)$ and c is the propagation velocity ($\mathbf{p}$ and t are the spatial and time variables in meters and in seconds, respectively). The time-dependent sound pressure is equal $Q(\mathbf{p},t)=-\rho\frac{\partial}{\partial t}\psi(\mathbf{p},t)$, where ρ is the density of the acoustic medium.

Transferring from the time domain to the frequency domain, the velocity potential ψ can be expressed as follows:

(3) $$\Psi(\mathbf{p},t)=\mathrm{Re}\{\varphi(\mathbf{p})\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}\},$$

where: $\omega=2\pi f$ [1/s] and $\varphi(\mathbf{p})$ is the complex velocity potential.

The substitution of the above expression into the wave equation reduces it to the Helmholtz equation of the form [3,5]:

(4) $$\nabla^2\varphi(\mathbf{p})+k^2\varphi(\mathbf{p})=0,$$

where $k^2=\frac{\omega^2}{c^2}$ and is the wave number.

The complex-valued function $\varphi(\mathbf{p})$ possess the magnitude and phase shift. The velocity vector has a similar form to Eq. (3): $\mathbf{v}(\mathbf{p},t)=\mathrm{Re}\{\nabla\varphi(\mathbf{p})e^{-\mathrm{i}\omega\mathrm{t}}\}$.

Often the normal component of the velocity on the boundary $v_n(\mathbf{p})$ is imposed as a boundary condition:

(5) $$v_n(\mathbf{p})=\nabla\varphi(\mathbf{p})\cdot\mathbf{n}_p=\frac{\partial\varphi(\mathbf{p})}{\partial n_p},$$

where $\mathbf{n}_p$ is the unit outward normal to the boundary at point $\mathbf{p}$ $[\mathrm{m}]$.

The sound pressure p at the point $\mathbf{p}$ in the acoustic region is one of the most useful acoustic properties, and it could be calculated as:

(6) $p(\mathbf{p}) = \mathrm{i}\omega\rho\varphi(\mathbf{p}) \left[\frac{1}{\mathrm{s}}\frac{\mathrm{kg}}{\mathrm{m}^3}\frac{\mathrm{m}^2}{\mathrm{s}} = \frac{\mathrm{kg}}{\mathrm{ms}^2} = \frac{\mathrm{N}}{\mathrm{m}^2} = \mathrm{Pa}\right].$

In practice, the magnitude of the sound pressure is measured on the decibel scale in which it is evaluated as the sound pressure level as $20\ \log_{10}\left|\frac{p(\mathbf{p})}{\sqrt{2}p^{ref}}\right|$, where p^{ref} is the reference pressure of 2×10^{-5} Pa.

The normal component of sound intensity $I(\mathbf{p})$ at points $\mathbf{p}$ on a boundary is expressed as follows:

(7) $$I(\mathbf{p}) = \frac{1}{2}\mathrm{Re}\{p^*(\mathbf{p})v_n(\mathbf{p})\}$$

where $p^*(\mathbf{p})$ represents the complex conjugate of sound pressure. The sound power W is a line integral of the sound intensity:

(8) $$W = \int_\Gamma\ I(\mathbf{q})\,\mathrm{d}\Gamma_q,$$

where Γ is a boundary line in a 2D problem.

Velocity potential definition

A velocity potential, introduced by Joseph-Louis Lagrange in 1788, is a scalar potential used in potential flow theory [4].

It is used in continuum mechanics when a continuum occupies a simply connected region (if the region has no holes - the such region is called connected) and is irrotational. In such a case:

(9) $$\nabla\times\mathbf{u} = 0\,,$$

where vector $\mathbf{u}$ [m/s] denotes the flow velocity. As a result, $\mathbf{u}$ can be represented as the gradient of a scalar function φ [2]:

(10) $$\mathbf{u} = \nabla\varphi\ = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k},$$

where φ [m^2/s] is known as a velocity potential for the flow velocity $\mathbf{u}$.

The velocity potential is not unique. If φ is a velocity potential, then is unique up to a constant [2].

Usage in acoustics

In theoretical acoustics [2], it is often desirable to work with the acoustic wave equation (2) converted to the Helmholtz equation (4) of the complex velocity potential φ instead of pressure p or velocity vector $\mathbf{u}$.

But, when φ is the complex solution of Eq. (4), not only $\mathbf{u}$ is found as given above but p could also be found from the linearised Bernoulli equation [2] for irrotational flow as it is indicated in Eq. (6).

Acoustic media

In acoustic parameters of the fluid varies with conditions such as the temperature of the medium and the pressure. However, in this paper, it will be assumed, for simplicity, that acoustic parameters would be independent of them. It will be considered the two most important acoustic media like air and water. Typically, air at 20°C and under one atmosphere of pressure has a density ρ =1.29 kg/m^3 and speed of sound c = 334 m/s. Water at 4°C has a density ρ =1000 kg/m^3 and a speed of sound c = 1524 m/s [13].

From Eq. (4) it can be deduced that $f = \frac{\mathrm{c}}{2\pi}k = 53.16k$ for air and $f = \frac{\mathrm{c}}{2\pi}k = 242.55k$ for water under the above conditions [13]. The sound pressure $p(\mathbf{p})$ is related to the velocity potential by Eq. (6).

For air:

(11) $$p(\mathbf{p}) = \mathrm{i}\ 430.86k\varphi(\mathbf{p}),$$

where 430.86 is the air acoustic impedance, and for water:

(12) $$p(\mathbf{p}) = \mathrm{i}\ 1524\cdot10^3k\varphi(\mathbf{p}).$$

where $1524\cdot10^3$ stands for the water acoustic impedance.

The formula for the wavelength is:

(13) $$\lambda = \frac{\mathrm{c}}{f}\ [\mathrm{m}]$$

An integral formulation for the forward acoustic problem

In acoustic problems described in a frequency domain (see Eq. (4)), integral formulation is:

(14) $$c(\boldsymbol{r})\varphi(\boldsymbol{r}) + \int_{\boldsymbol{\Gamma}} \frac{\partial G(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial n}\varphi(\boldsymbol{r}')d\boldsymbol{\Gamma} = \int_{\boldsymbol{\Gamma}}\ G(|\boldsymbol{r}-\boldsymbol{r}'|)\frac{\partial\varphi(\boldsymbol{r}')}{\partial n}d\boldsymbol{\Gamma}$$

Now the integral boundary equation (14) for constant boundary elements can be written in terms of local coordinate ξ instead of the boundary line $\boldsymbol{\Gamma}$, as follows:

(15) $$c(\boldsymbol{r})\varphi(\boldsymbol{r}) + \sum_{j=1}^{M}\varphi_j(\boldsymbol{r}')\int_{-\mathbf{1}}^{+1}\frac{\partial G(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial n}J(\xi)d\xi = \sum_{j=1}^{M}\frac{\partial\varphi_j(\boldsymbol{r}')}{\partial n}\int_{-1}^{+1}G(|\boldsymbol{r}-\boldsymbol{r}'|)J(\xi)d\xi$$

where M – is the total number of constant elements, and $J(\xi)$ – is the Jacobian of transformation (see Eq. (16)).

(16) $$J(\xi) = \sqrt{\left(\frac{x_1-x_0}{2}\right)^2 + \left(\frac{y_1-y_0}{2}\right)^2} = \frac{1}{2}L$$

where (x_0, y_0) and (x_1, y_1) are the edge points and L is the length of the constant element.

The functions under an integral sign which contain the kernels can be substituted by the functions $A_{i,j}$ and $B_{i,j}$ as follows:

(17) $$c(\boldsymbol{r})\varphi(\boldsymbol{r}) + \sum_{j=1}^{M}\varphi_j(\boldsymbol{r}')\,A_{i,j}(\boldsymbol{r},\boldsymbol{r}') = \sum_{j=1}^{M}\frac{\partial\varphi_j(\boldsymbol{r}')}{\partial n}B_{i,j}(\boldsymbol{r},\boldsymbol{r}')$$

To form a set of linear algebraic equations, we take each node in turn as a load point $\boldsymbol{r}$ and perform the integrations indicated in Eq. (15). This will result in the following system of algebraic equations in matrix form:

$$[\boldsymbol{A}][\boldsymbol{\varphi}] = [\boldsymbol{B}]\left[\frac{\partial \boldsymbol{\varphi}}{\partial n}\right] \tag{18}$$

where the matrices $[\boldsymbol{A}]$ and $[\boldsymbol{B}]$ contain the integrals of the kernel's normal derivative $\frac{\partial G(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial n}$ and the kernels $G(|\boldsymbol{r}-\boldsymbol{r}'|)$ respectively, i.e., the functions $A_{i,j}$ and $B_{i,j}$ of Eq. (17).

In acoustic problems described by Eq. (4) G is the free space Green function [13,15]:

$$G(|\boldsymbol{r}-\boldsymbol{r}'|) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|\boldsymbol{r}-\boldsymbol{r}'|) \tag{19}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero [1] and imaginary unit $\mathrm{i} = \sqrt{-1}$ and $r = |\boldsymbol{r}-\boldsymbol{r}'|$.

Therefore, the kernel derivative with respect the normal direction in the collocation point can be expressed:

$$\frac{\partial G(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial n} = -\frac{\mathrm{i}}{4} k H_1^{(1)}(k|\boldsymbol{r}-\boldsymbol{r}'|)\left(\frac{x-x'}{r} n_x + \frac{y-y'}{r} n_y\right) \tag{20}$$

where $H_1^{(1)}$ is the spherical Hankel function of the first kind and order one.

Singular integration for acoustic problems for constant element

As it was shown in previous paragraphs, the singular integration is a severe problem for the Boundary element, and it has a considerable influence on the final precision. The more complicated problem, the more sensitive the singularity is.

For the acoustic problems, an extensive review of singular integration is provided by S.M. Kirkup et al. in excellent work [13]. The authors presented several methods like:

1. ignoring the singularity,
2. product integration,
3. subtracting out the singularity,
4. substitution or transformation.

All methods are remarkably interesting, but for the inverse problems in acoustic, the subtracting out singularity method seems to be the most appropriate.

This method is widely applied in BEM code and provides a high-precision solution. This method relies on splitting the integrand into two parts the singular part and the regular one. For example, if $f(x) \sim \psi(x)$ near to the singularity, then writing $f(x) = \psi(x) + [f(x) - \psi(x)]$ the last term in a square bracket may be a regular one. The singularity does not disappear but $\int \psi(x)\, dx$ in some cases (for example, see the Laplace's equation) it could be calculated analytically and might be used in many other problems.

A particularly good example is the Helmholtz equation where the singularity could be subtracted in the following way:

$$\int_{\boldsymbol{\Gamma}} G(|\boldsymbol{r}-\boldsymbol{r}'|)\, d\boldsymbol{\Gamma} = \int_{\boldsymbol{\Gamma}} G_0(|\boldsymbol{r}-\boldsymbol{r}'|) d\boldsymbol{\Gamma} + \int_{\boldsymbol{\Gamma}} [G(|\boldsymbol{r}-\boldsymbol{r}'|) - G_0(|\boldsymbol{r}-\boldsymbol{r}'|)] d\boldsymbol{\Gamma} \tag{21}$$

For 2D problems exists the analytical expression for the integral $\int_{\boldsymbol{\Gamma}} G_0(|\boldsymbol{r}-\boldsymbol{r}'|) d\boldsymbol{\Gamma}$, then for acoustic problems described by the Helmholtz equation we will get:

$$\begin{aligned} G(|\boldsymbol{r}-\boldsymbol{r}'|) - G_0(|\boldsymbol{r}-\boldsymbol{r}'|) &= \frac{\mathrm{i}}{4} H_0^{(1)}(kr) - \frac{-1}{2\pi}\ln(r) \\ &= \frac{\mathrm{i}}{4}\left(J_0(kr) + iY_0(kr)\right) + \frac{1}{2\pi}\ln(r) \\ &= -\frac{1}{4} Y_0(kr) + \frac{1}{2\pi}\ln(r) + \frac{\mathrm{i}}{4} J_0(kr) \end{aligned} \tag{22}$$

where: J_0 and Y_0 are the Spherical Bessel functions of the first and the second kind, respectively [1].

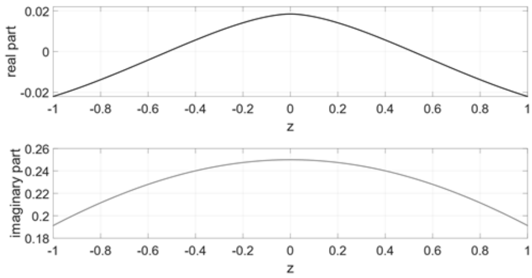

Fig. 2. Singular integrand after subtracting out procedure the real part $-\frac{1}{4} Y_0(|z|) + \frac{1}{2\pi}\ln(|z|)$ above and the imaginary part $\frac{1}{4} J_0(|z|)$ below

The above figures present that integrand (Eq. 22), after the subtracting out procedure is now the regular function. So, for numerical calculation, the standard Gaussian quadrature could be used.

The first integral in Eq. (21) $\int_{\boldsymbol{\Gamma}} G(|\boldsymbol{r}-\boldsymbol{r}'|)\, d\boldsymbol{\Gamma} = \int_{\boldsymbol{\Gamma}} G_0(|\boldsymbol{r}-\boldsymbol{r}'|) d\boldsymbol{\Gamma} +$ has the singularity, but following the Laplace equation in 2D space, it could be calculated analytically (see, for example [1]).

Analytical solution of the acoustic problem

To solve more advanced acoustic problems, let us start with the excellent benchmark problem suggested in [13].

a) b)

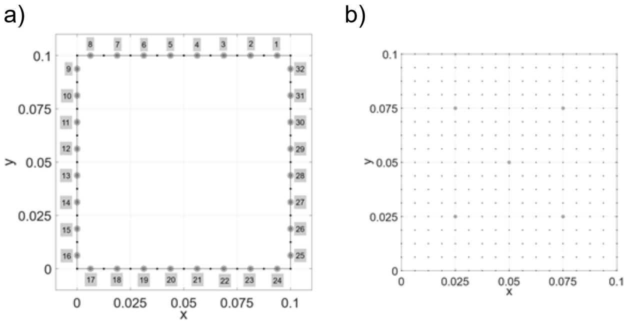

Fig. 3. Region of interest a) discretization b) internal point distribution with control points

Let us consider the acoustic environment is air at 20°C and the pressure is one atmosphere (symbol atm is equal to 101.325kPa). The speed of sound is equal to 344m/s and the frequency of the test is 400Hz, hence $k = 7.31\ m^{-1}$ and the wavelength $\lambda = \frac{c}{f} = 0.86$m. Inside the domain which is the interior of a square of side 0.1m distribution of the velocity potential φ is equal to:

$$\varphi(\mathbf{p}) = \sin\left(\frac{k}{\sqrt{2}} x\right)\sin\left(\frac{k}{\sqrt{2}} y\right) \tag{23}$$

The above equation is the solution of the Helmholtz equation (see Eq. (4) with the boundary conditions Eq. (24)

and Eq. (25)). On one part of the boundary, the homogeneous Dirichlet boundary conditions are imposed:

$$\varphi(\mathbf{p}) = \begin{cases} 0 & \text{when } x = 0 \\ 0 & \text{when } y = 0 \end{cases} \quad (24)$$

and on the other nonhomogeneous Dirichlet boundary condition of the following form:

$$\varphi(\mathbf{p}) = \begin{cases} \sin\left(\frac{k}{\sqrt{2}}0.1\right)\sin\left(\frac{k}{\sqrt{2}}y\right) & \text{when } x = 0.1 \\ \sin\left(\frac{k}{\sqrt{2}}x\right)\sin\left(\frac{k}{\sqrt{2}}0.1\right) & \text{when } y = 0.1 \end{cases} \quad (25)$$

For Helmholtz equation with the above Dirichlet boundary conditions solution in the form of the absolute value of potential velocity surface and in the form of equipotential lines are presented in figure 4:

a) b)

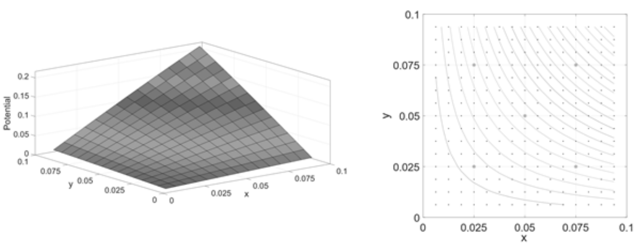

Fig. 4. Region of interest a) potential surface over the region b) equipotential lines distribution and the fifth control points depicted by circular markers

Having the analytical solution after numerical calculation, it will be possible to control the error of the numerical solution in the control points indicated in Fig. 4b.

Two cases were considered when the side of the region was 0.1m, and the next inside the region was ten times bigger up to 1m. The maximal values of the kernel arguments are equal to 0.039 and 0.39, respectively.

Comparison of the numerical solution with the analytical one in the fifth internal point shown in Fig. 4b is assembled in table 1.

a) b)

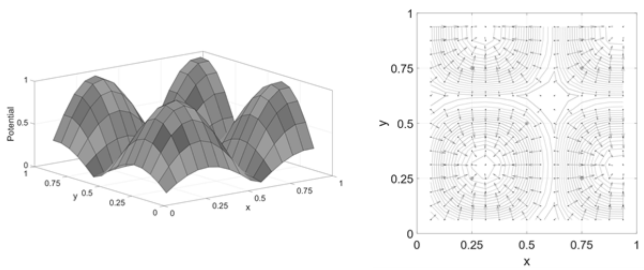

Fig. 5. Region of interest a) potential surface over the region b) equipotential lines and the velocity vector distribution

Table 1. Relative error of calculations

Control points coordinates	Error [%] for square length 0.1m	Error [%] for square length 1m
(0.025,0.025)	0.0041	1.29
(0.075,0.025)	0.0128	0.51
(0.025,0.075)	0.0128	0.51
(0.075,0.075)	0.0054	1.06
(0.050,0.050)	0.0083	1.04

Conclusions

In this paper, the influence of singularity integration on the precision of calculation of the forward acoustic problems was presented. The acoustic problem was described by the Helmholtz equation in 2D space. As a state variable, the velocity potential was selected. Analysis was conducted in the frequency domain. Singular integrands were calculated by subtracting out the singular term, which next could be calculated analytically. It was proved that such an approach provides high-precision results formulated for acoustic problems, even for the coarse discretization of the boundary. See, for example, table 1 and Fig. 3. For calculations, the benchmark with the analytical solution proposed in [13] was used to prove the effectiveness and precision of such an approach.

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