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Mathematical models of electric arc with variable length using column radius as a state variable

Modele matematyczne łuku elektrycznego o zmiennej długości z wykorzystaniem promienia kolumny jako zmiennej stanu

Abstract: *Practical justification of research aimed at creating mathematical models of arc with variable length of plasma column is given. A model with geometric radius of the arc as a state variable was developed, allowing for taking into account the dynamics of changes in column length. Simulation studies confirmed the usefulness of such mathematical model.*

Streszczenie. *Podano praktyczne uzasadnienie badań zmierzających do tworzenia modeli matematycznych łuku o zmiennej długości kolumny plazmowej. Opracowano model z promieniem geometrycznym łuku jako zmienną stanu umożliwiający uwzględnienie dynamiki zmian długości kolumny. Badania symulacyjne potwierdziły użyteczność takiego modelu matematycznego.*

Keywords: electric arc, mathematical model, arc radius as a state variable, variable arc length
Słowa kluczowe: łuk elektryczny, model matematyczny, promień łuku zmienną stanu, zmienna długość łuku

Introduction

Changes in the length of the arc column are often used in the processes of regulating the thermal power of electrothermal and welding devices. Depending on the technological needs, these changes can take place in different length ranges and at different speeds. Certain limitations to these changes exist due to the possibility of losing stability and extinguishing the arc.

In some technological processes [1], changes in the length of the column are accompanied by changes in the position of the electrode spots. Therefore, sometimes the type of electrode material, its temperature and shape affect the possibilities of electric arc displacement. Then the changes in the length of the column can even be abrupt.

Mechanical regulation of the length of the column is the basic method of controlling the thermal power of three-phase arc furnaces [2, 3, 4]. Despite such controlled electrode displacements, various electromagnetic and gas-dynamic factors occur that simultaneously disturb the set arc lengths.

Mechanical changes in the length of the arc are often used in welding. In addition to manual or automatic electrode displacement, various external disturbances are also possible here, affecting the actual length of the column. After all, the quite popular phenomenon of magnetic blast is known.

The characteristic feature of mechanical interactions causing changes in arc length is their relatively low speed and low frequency. In such cases, classical mathematical models of the arc are often sufficient. The variation of constant parameters is introduced in them, which sometimes allows obtaining satisfactory results of simulation of processes in circuits with arcs [1].

Magnetic interactions on the arc column do not have such speed or frequency restrictions. However, it should be remembered that the arc has a certain small inertia, which is represented in linear mathematical models using a time constant. Too rapid changes in length in relation to the inertia of the arc can affect its stability. Magnetic exciters with metal cores also have their own inertia, which affects the limitation of the frequency of changes in arc length.

According to the method proposed by Voronin, modifying the Mayr and Cassie models allows for proper representation of physical processes in low and high power arcs with variable column length [5]. Then the basic state variable is the conductance or resistance of the plasma column. Combining these two models (low-current and high-current) into a hybrid model further expands the possibilities of its use in simulations of the operating states of welding and electrothermal systems.

An alternative to the classical mathematical Mayr and Cassie models may be the model of an arc with a state variable being the geometric radius of a cylindrical plasma column [6]. Moreover, it is possible to modify this model so that it is useful for representing arcs with a variable column length. This will enable its wider use in research carried out by electricians, welders and electrometallurgists. Exemplary analyses of systems with different models of variable-length arcs are considered in the publications [7-10].

Mathematical model of a variable length arc column

Usually, the starting assumption for developing a mathematical model of an arc column is the energy balance. It can be written in the form of an energy flow balance

(1) $$p_1(t)+p_2(t)=p_3(t)$$

where: p_1 - dissipated heat power, W; p_2 - heat power stored in the arc column, W; p_3 - electrical power supplied, W.

The dissipated power from the arc p_1 depends on the plasma temperature T. If the arc is high-current, the characteristics of highly heated gases are weakly dependent on changes in its high temperature. Therefore, this dependence is omitted. However, the following approximation of the dependence of the power dissipated from the side surface of the column on its parameters is assumed [11], i.e. on the radius r and the length of the arc column l

(2) $$p_1(t)=k_1 r^n l^k$$

If the ambient temperature is high, the heat transfer from the arc is inefficient and then $p_1=f(r,l)=k_1 r^0 l^0=const$ and from here n = 0, k = 0;

If the arc is long, then $p_1=f(S_b)$, where S_b – lateral surface area of the arc column. It follows that $p_1=f(r,l)=k_1 r^1 l^1$, and from here n = 1; k = 1.

If the arc is short, then $p_1 = f\left(S_p\right) = k_1 r^2$, where S_p – cross-sectional area of the arc column (also the electrode spot). It follows that n = 2 and k = 0.
In the situations considered, we can write down the relationship between the exponents of the power function $p_1(t)$ in the form of $k = -(n-1)^2 + 1$.

The power of energy stored in the column is described by the relationship

(3) $$p_2(t) = \frac{dQ}{dt}$$

where Q – plasma enthalpy, J. This value depends on the volume of the column $Q \propto V \propto r^2 l$, thus it is given with

(4) $$p_2(t) = k_2\left(2r\frac{dr}{dt}l + r^2\frac{dl}{dt}\right)$$

The electrical power supplied to the arc column can be described by the relationship

(5) $$p_3(t) = ui = r_a i^2$$

where r_a – instantaneous resistance of the arc column, Ω. The resistance of the column is given by the formula

(6) $$r_a = \frac{\rho(r)l}{S(r)}$$

where: l – arc length, m; S – cross-sectional area of the column, m^2; ρ – plasma resistivity, Ωm. The latter quantity is non-linearly dependent on r [7]

(7) $$\rho(r) \approx r^{-m}$$

If we assume the dependence $S = \pi r^2$, we will get

(8) $$p_3(t) = \frac{\frac{k_3 l}{r^m}}{r^2}i^2 = \frac{k_3 l}{r^{m+2}}i^2$$

The general form of the mathematical model of the arc is a nonlinear ordinary differential equation

(9) $$k_1 r^n l^k + k_2\left(2r\frac{dr}{dt}l + r^2\frac{dl}{dt}\right) = \frac{k_3 l}{r^{m+2}}i^2$$

The voltage value on the arc column can be calculated from the relationship

(10) $$u = \frac{i}{g} = r_a i = \frac{k_3 l}{r^{m+2}}i$$

Hence, the conductance can be expressed as a function of the column radius

(11) $$g = \frac{r^{m+2}}{k_3 l}$$

or vice versa the column radius as a function of conductance

(12) $$r = (k_3 g l)^{\frac{1}{m+2}}$$

Formula (9) can be written in a more convenient form

(13) $$\frac{dr}{dt} = \frac{k_3}{2k_2}\frac{i^2}{r^{m+3}} - \frac{k_1}{2k_2}r^{n-1}l^{k-1} - \frac{1}{2}\frac{r}{l}\frac{dl}{dt}$$

The basis for creating the arc column macromodel is the integrated equation (13)

(14) $$r = \int_0^t\left(\frac{k_3}{2k_2}\frac{i^2}{r^{m+3}} - \frac{k_1}{2k_2}r^{n-1}l^{k-1} - \frac{1}{2}\frac{r}{l}\frac{dl}{dt}\right)d\tau + r_0$$

If we now rearrange formula (14), we can calculate the instantaneous value of g or r_a. The conductance can be calculated from the formula

(15) $$g = \frac{r^{m+2}}{k_3 l} = \frac{1}{k_3 l}\left\{\int_0^t\left(\frac{k_3}{2k_2}\frac{i^2}{r^{m+3}} - \frac{k_1}{2k_2}r^{n-1}l^{k-1} - \frac{1}{2}\frac{r}{l}\frac{dl}{dt}\right)d\tau + r_0\right\}^{m+2}$$

Then, it is convenient to use a controlled voltage source when creating a macromodel.

In some calculations of circuits with electric arcs, macromodels are created using a controlled current source [1, 5]. To do this, formula (13) should be transformed into the form

(16) $$\frac{1}{r}\frac{dr}{dt} = \frac{d(\ln r)}{dt} = \frac{k_3}{2k_2}\frac{i^2}{r^{m+4}} - \frac{k_1}{2k_2}r^{n-2}l^{k-1} - \frac{1}{2l}\frac{dl}{dt}$$

Then, after further transformations, an integral equation can be obtained from it

(17) $$r = r_0\exp\left[\int_0^t\left(\frac{k_3}{2k_2}\frac{i^2}{r^{m+4}} - \frac{k_1}{2k_2}r^{n-2}l^{k-1} - \frac{1}{2l}\frac{dl}{dt}\right)d\tau\right]$$

Based on (12) the conductance value can be determined

(18) $$g = \frac{r^{m+2}}{k_3 l} = \frac{1}{k_3 l}\left\{r_0\exp\left[\int_0^t\left(\frac{k_3}{2k_2}\frac{i^2}{r^{m+4}} - \frac{k_1}{2k_2}r^{n-2}l^{k-1} - \frac{1}{2l}\frac{dl}{dt}\right)d\tau\right]\right\}^{m+2}$$

If the electric arc has a constant length, then the increment values dl/dt = 0 and the above formulas are simplified to the form described in the publication [6].

Equation (16) contains two derivatives that can be integrated simultaneously to obtain a more advantageous formula

(19) $$r^2 l = r_0^2 l_0 \exp\left[\int_0^t\left(\frac{k_3}{k_2}\frac{i^2}{r^{m+4}} - \frac{k_1}{k_2}r^{n-2}l^{k-1}\right)d\tau\right]$$

Then we can obtain a simpler integral equation for the arc conductance from it

$$g = \frac{r^{m+2}}{k_3 l} = \qquad (20)$$
$$= \frac{1}{k_3 l}\left\{ \frac{r_0^2 l_0}{l} \exp\left[\int_0^t \left(\frac{k_3}{k_2} \frac{i^2}{r^{m+4}} - \frac{k_1}{k_2} r^{n-2} l^{k-1} \right) d\tau \right] \right\}^{\frac{m+2}{2}}$$

where r_0 and l_0 are the initial conditions for solving the differential equation.

Simulation test results of the variable-length electric arc mode

In addition to the voltage drop along the plasma column, additional near-electrode voltage drops occur in electric arcs. It is assumed that their values do not depend on the current intensity, and therefore are constant for the same electrodes. These voltage drops depend on the electrode materials, their design, additional heating or cooling, etc. Voltage drops near the anode are usually smaller than near the cathode. If short (low-voltage) arcs are modeled, the sum of the values of voltage drops near the electrode is comparable to the voltage on the plasma column and then it should be taken into account. In the case of modeling long (high-voltage) arcs, voltage drops near the electrodes are relatively small and then they can be ignored. In the calculations performed, a constant value of the sum of voltages near the electrodes was assumed, independent of the polarization U_{AC} = 15 V.

The mathematical models developed (15), (18) and (20) of the arc with a variable column length were used to build three macromodels. Each of them was separately checked by simulation method. Figures 1, 2 and 3 show families of dynamic characteristics of electric arc with selected constant lengths l. Loops have different colors depending on the length of arc column. In captions under figures, letter Bl indicates navy blue, letter Gr indicates green, and letter Re indicates red. Arc is powered from sinusoidal current source with parameters $I_{\max}$ = 300 A, f = 50 Hz. Difference between arc models also lies in the value of exponent m. It is related to plasma resistivity (7). Resistivity ρ decreases with increasing exponent m, which corresponds to increasing plasma temperature.

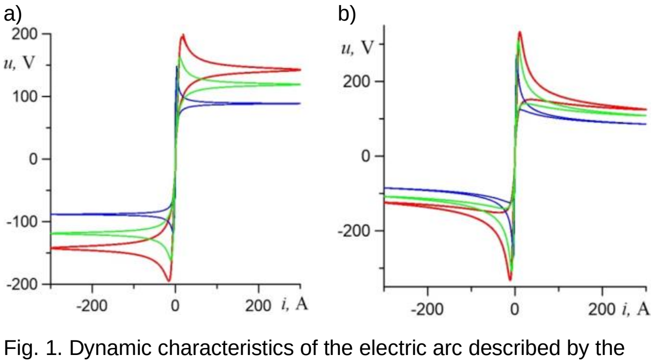

Fig. 1. Dynamic characteristics of the electric arc described by the model (15) with different column lengths (k_1 = 18000, k_2 = 100, k_3 = 30, n = 2, Bl - (l = 0,01 m), Gr - (l = 0,02 m), Re - (l = 0,03 m)): a) m = 0, b) m = 1

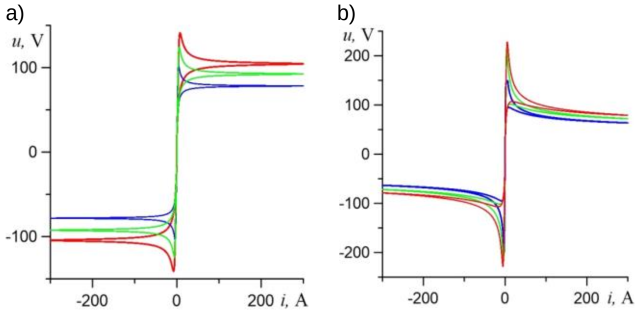

Fig. 2. Dynamic characteristics of the electric arc described by the model (18) with different column lengths (k_1 = 5000, k_2 = 15, k_3 = 30, n = 2, Bl - (l = 0,01 m), Gr - (l = 0,02 m), Re - (l = 0,03 m)): a) m = 0, b) m = 1

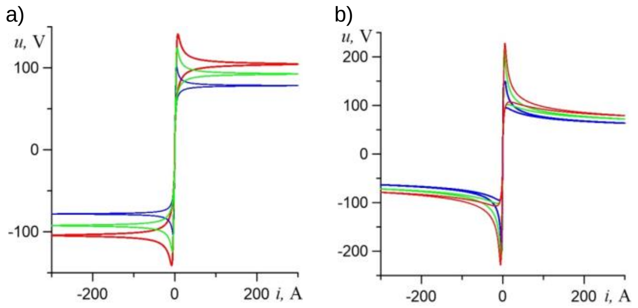

Fig. 3. Dynamic characteristics of the electric arc described by the model (20) with different column lengths (k_1 = 5000, k_2 = 5, k_3 = 20, n = 2, Bl - (l = 0,01 m), Gr - (l = 0,02 m), Re - (l = 0,03 m)): a) m = 0, b) m = 1

It was assumed that this change is non-uniform and takes place according to the formula

$$l = At + B + 0.5\{\tanh[C(t-D)]+1\}E \qquad (21)$$

where: A, B, C, D, E – constant coefficients of approximation of the length course. Examples of arc length changes are shown in Figure 4.

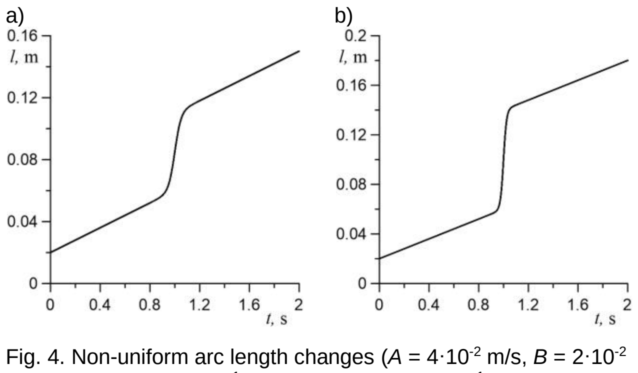

Fig. 4. Non-uniform arc length changes ($A = 4\cdot10^{-2}$ m/s, $B = 2\cdot10^{-2}$ m, D = 1 s): a) C = 20 s^{-1}, E = 0.05 m; b) C = 40 s^{-1}, E = 0.08 m

Figures 5 and 6 show the result of the process of deforming the dynamic characteristics of the arc under the influence of changes in the length of the plasma column.

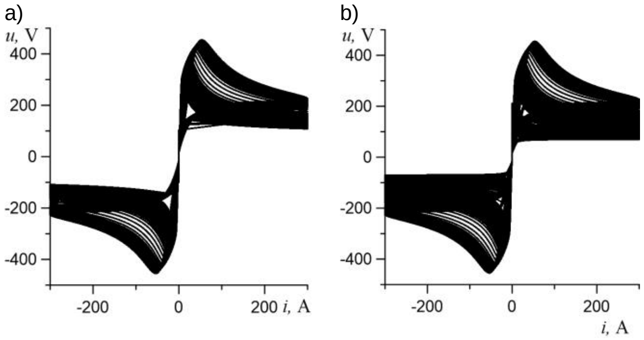

Fig. 5. Dynamic characteristics of the stretched electric arc (n = 2, A = 4·10^{-2} m/s, B = 2·10^{-2} m, D = 1 s): a) described by the model (15) (k_1 = 18000, k_2 = 100, k_3 = 30, m = 1, C = 20 s^{-1}, E = 0,05 m), b) described by the model (18) (k_1 = 5000, k_2 =15, k_3 = 30, m = 0, C = 40 s^{-1}, E = 0,08 m)

The parameters of the arc models were assumed as in the cases of Figures 1, 2 and 3, except for the variable value of length l. The simulation time was 2 s and in the middle of this time there was a rapid increase in length. As a result, the most densely filled areas are obtained at the beginning and at the end of the simulation, while the center is the least filled. Figures 5 and 6 illustrate the effect of assumed $l(t)$ dependence on the dynamical $u(i)$ dependencies. Model (20) does not contain a derivative of length and therefore it can change abruptly.

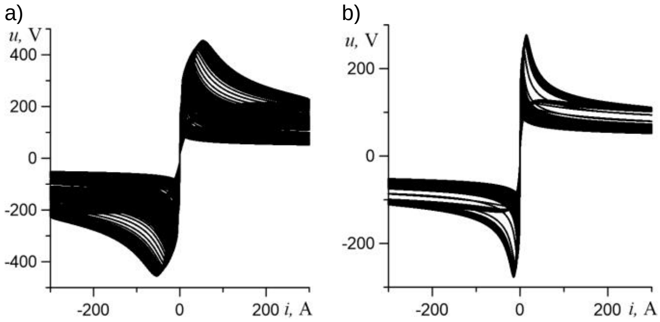

Fig. 6. Dynamic characteristics of the stretched electric arc described by the model (20) (m = 1, n = 2, k_1 = 5000, k_2 = 5, k_3 = 20, A = 4·10^{-2} m/s, B = 2·10^{-2} m, D = 1 s): a) C = 30 s^{-1}, E = 0,06 m, b) C = 50 s^{-1}, E = 0,12 m)

In order to compare the compatibility of solutions of two different models (18) and (20), simulation tests were performed with identical preset parameters. The results are shown in Figure 7. After a close look, it can be seen that the differences between them are small and concern the area of current passing through the zero value.

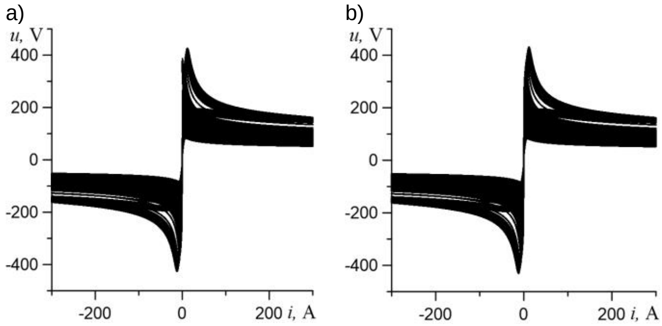

Fig. 7. Dynamic characteristics of the stretched electric arc (m = 1, n = 2, A = 5·10^{-2} m/s, B = 2·10^{-2} m, C = 50 s^{-1}, D = 1 s, E = 0,06 m, k_1 = 10000, k_2 = 10, k_3 = 25): a) described by the model (18), b) described by the model (20)

Conclusions:

1. Using the methodology of creating a mathematical model of an electric arc with the geometric radius of the column as a state variable, it was extended to obtain a model with a variable arc length.
2. The differential and integral mathematical forms of the new variable-length arc models are relatively simple, which allows the creation of computer macromodels using voltage or current controlled sources.
3. Computer simulations of the new arc models have shown the possibilities of mapping the characteristics of an arc with a time-varying current excitation and with a variable arc column length.
4. The developed mathematical models extend the set of available variable-length arc models.

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