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Angles between transient values in the linear electrical circuits

Kąty pomiędzy stanami nieustalonymi w liniowych obwodach elektrycznych

Abstract: *The classical definition of the angles between steady values of the voltages and currents in the electrical circuits with constant values of the resistances, inductances and capacitances will be extended to the transient values of the voltages and currents. The theses will be shown on simple electrical circuits with constant resistances, inductances and capacitances. It will be shown that the angles depend on the location of the resistances, inductances and capacitances of the electrical circuits.*

Streszczenie: *Klasyczna definicja kątów między wartościami ustalonymi napięć i prądów w obwodach elektrycznych o stałych wartościach rezystancji, indukcyjności i pojemności zostanie rozszerzona na stany przejściowe napięć i prądów. Tezy zostaną przedstawione na prostych obwodach elektrycznych o stałych wartościach rezystancji, indukcyjności i pojemności. Zostanie pokazane, że kąty zależą od położenia rezystancji, indukcyjności i pojemności obwodów elektrycznych.*

Keywords: angle, transient states, linear, electrical circuit.
Słowa kluczowe: kąt, stany przejściowe, liniowy, obwód elektryczny.

Introduction

In the classical linear circuits with sinusoidal inputs the notion of the angle between voltages and currents is well defined and used for the sinusoidal voltages and currents [1, 2, 5, 8, 11].

In this paper it will be shown that the notion of angles can be extended to the transient values of the voltages and currents in the linear electrical circuits with constant values of resistances, inductances and capacitances.

The paper is organized as follows. In Section 2 some basic definitions from the functional analysis concerning the scalar product of vectors and matrices and the angles between vectors and matrices are recalled. The angles between currents and voltages in the linear electrical circuits in transient state are defined in Section 3 and illustrated by simple electrical circuits with constant parameters. The dependence of the angles on the locations of the resistances, inductances and capacitances is analyzed in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of $n\times m$ real numbers, $\Re^{n\times m}$ – the set of real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, I_n - the $n\times n$ – the identity matrix.

Basic definitions

It is well-known [3–8, 10, 11] that in the *n*-dimensional real vector space $\Re^n$ the scalar product is well defined if for every pair of vectors $x,y\in\Re^n$ there exists a real number (x,y) satisfying the following conditions:

1. The scalar product is symmetric (*x*, *y*) = (*y*, *x*) for all $x,y\in\Re^n$
2. $(\lambda x,y)=\lambda(y,x)$ for every real number λ and all $x,y\in\Re^n$
3. $(x_1+x_2,y)=(x_1,y)+(x_2,y)$ for all $x_1,x_2\in\Re^n$
4. $(x,x)>0$ and $(x,x)=0$ if and only if $x=0$

The scalar product of real vectors $x=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$, $y=\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}$ in orthogonal basis is given by

$$(x,x)=\sum_{i=1}^n x_i y_i\,. \tag{2.1}$$

The module of the vector $x\in\Re^n$ is given by

$$|x|=\sqrt{(x,x)} \tag{2.2}$$

and the angle between the vectors *x* and *y* is defined by

$$\varphi=arc\cos\frac{(x,y)}{|x||y|},\quad 0\le\varphi\le\pi\,. \tag{2.3}$$

In Euclides space of continuous functions $f(t),g(t)$ the scalar product is given by

$$(f(t),g(t))=\int_a^b f(t)g(t)dt,\ a,\ b-\text{given numbers} \tag{2.4}$$

and the angle between $f(t)$ and $g(t)$ is defined by

$$\varphi=\varphi(\infty)=arc\cos\frac{\int_0^\infty f(t)g(t)dt}{\sqrt{\int_0^\infty [f(t)]^2dt}\sqrt{\int_0^\infty [g(t)]^2dt}},\quad 0\le\varphi\le\pi, \tag{2.5a}$$

$$\varphi(t)=arc\cos\frac{\int_0^t f(t)g(t)d\tau}{\sqrt{\int_0^t [f(t)]^2d\tau}\sqrt{\int_0^t [g(t)]^2d\tau}},\quad 0\le t\le\infty. \tag{2.5b}$$

The definitions (2.4) and (2.5) can be extended to the vectors

$$f(t)=\begin{bmatrix}f_1(t)\\ \vdots\\ f_n(t)\end{bmatrix},\quad g(t)=\begin{bmatrix}g_1(t)\\ \vdots\\ g_n(t)\end{bmatrix} \tag{2.6}$$

and

(2.7) $$(f(t),g(t))=\int_a^b\sum_{i=1}^n f_i(t)g_i(t)dt\,.$$

The angle between the vectors functions (2.6) is defined by

$$\varphi=\varphi(\infty)=arc\cos\frac{\int_0^\infty\sum_{i=1}^n f_i(t)g_i(t)dt}{\sqrt{\int_0^\infty\sum_{i=1}^n f_i^2(t)dt}\sqrt{\int_0^\infty\sum_{i=1}^n g_i^2(t)dt}},\quad 0\le\varphi\le\pi,$$
(2.8a)

$$\varphi(t)=arc\cos\frac{\int_0^t\sum_{i=1}^n f_i(t)g_i(t)d\tau}{\sqrt{\int_0^t\sum_{i=1}^n f_i^2(t)d\tau}\sqrt{\int_0^t\sum_{i=1}^n g_i^2(t)d\tau}},\quad 0\le\varphi(t)\le\pi.$$
(2.8b)

The notion of the angle can be also extended to matrices with entries depending on time t as follows [2, 4, 10]. Let's take two any given matrix $A=[a_{ij}]\in\Re^{n\times m}$, the following two corresponding vectors can be defined

$$\overline{A}=[a_{11}\ \ \dots\ \ a_{1n}\ \ a_{21}\ \ \dots\ \ a_{2n}\ \ a_{31}\ \ \dots\ \ a_{nm}]^T\in\Re^{n\times m}$$
(2.9a)

and

$$\hat{A}=[a_{11}\ \ \dots\ \ a_{n1}\ \ a_{12}\ \ \dots\ \ a_{n2}\ \ a_{13}\ \ \dots\ \ a_{nm}]^T\in\Re^{n\times m}$$
(2.9b)

where T denotes the transpose. The scalar

(2.10) $$(\overline{A},\overline{B})=(\hat{A},\hat{B})=\sum_{i=1}^n\sum_{j=1}^m a_{ij}b_{ij}$$

is called the scalar product of the matrices A and B. The angle defined by

$$\varphi=\varphi_{A,B}=arc\cos\frac{(\overline{A},\overline{B})}{|\overline{A}||\overline{B}|}=arc\cos\frac{(\hat{A},\hat{B})}{|\hat{A}||\hat{B}|},\quad 0\le\varphi\le\pi$$
(2.11)

is called the angle between the matrices A and B. From (2.11) it follows that

(2.12) $$\cos\varphi_{A,B}=\cos\varphi_{B,A},\quad \cos\varphi_{A,B}=\cos\varphi_{-A,-B}$$

and if $\overline{A}=\overline{B}$ then $\cos\varphi_{A,B}=1,\ \ \varphi_{A,B}=0$. The matrix defined by

$$A\circ B=\begin{bmatrix}a_{11}b_{11} & \dots & a_{1m}b_{1m}\\ \vdots & \dots & \vdots\\ a_{n1}b_{n1} & \dots & a_{nm}b_{nm}\end{bmatrix}\in\Re^{n\times m}$$
(2.13)

Is called the Hadamard product of the matrices A and B.

Note that the angle $\varphi_{A,B}$ between the matrices $\underline{A}$ and B satisfies the condition if and only if and if and only if $(\overline{A},\overline{B})\ge 0$ and $\cos\varphi_{A,B}<0$ if and only if $(\overline{A},\overline{B})<0$.

Note that the definition (2.8) can be extended to matrices with entries depending on time t.

Angles between currents and voltages in linear electrical circuits

We begin with simple linear electrical circuit shown in Fig. 3.1 with given resistance R and capacitance C and source voltage E.

The voltage $u=u_c(t)$ on the capacitance and current $i=i(t)$ in the circuit are given by

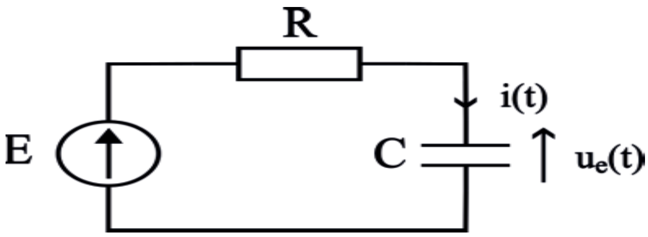

Fig. 3.1. Linear electrical circuit R, C.

(3.1) $$u=E(1-e^{-\frac{t}{RC}}),\quad i=C\frac{du}{dt}=\frac{E}{R}e^{-\frac{t}{RC}}$$

and shown in Fig. 3.2.

Using the definition (2.5) and (3.1) we obtain

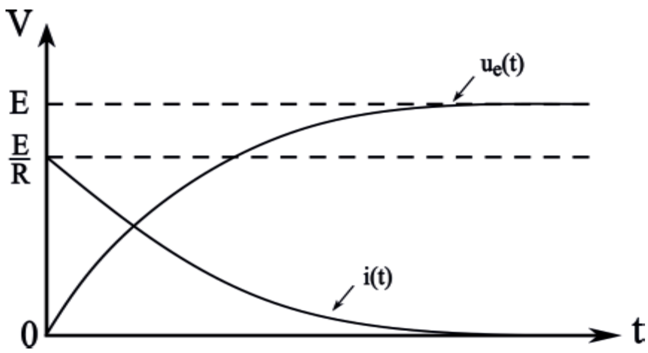

Fig. 3.2. Voltage *u(t)* and current *i(t)*.

$$\cos\varphi(t)=\frac{\int_0^t E(1-e^{-\frac{t}{RC}})\frac{E}{R}e^{-\frac{t}{RC}}d\tau}{\sqrt{\int_0^t E^2(1-e^{-\frac{t}{RC}})^2dt\tau}\sqrt{\int_0^t\left(\frac{E}{R}\right)^2\left(e^{-\frac{t}{RC}}\right)^2d\tau}}=$$

$$=\frac{\int_0^\infty(1-e^{-\frac{t}{RC}})e^{-\frac{t}{RC}}dt}{\sqrt{\int_0^\infty(1-e^{-\frac{t}{RC}})^2dt}\sqrt{\int_0^\infty\left(e^{-\frac{t}{RC}}\right)^2dt}}=0$$
(3.2)

since $\int_0^\infty(1-e^{-\frac{t}{RC}})^2dt=\infty$

Therefore, the angle between the voltage *u(t)* and the current *i(t)* is equal to $-\frac{\pi}{2}$ for $t=\infty$.

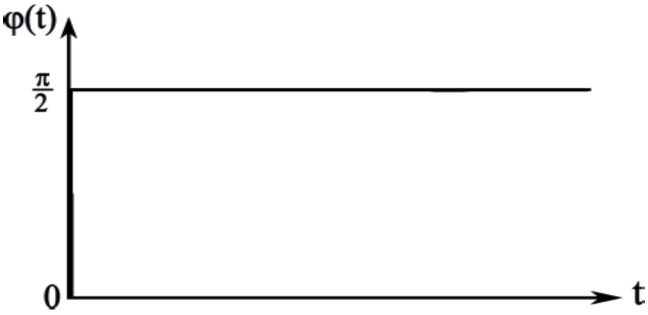

Fig. 3.3. The angle between voltage *u(t)* and current *i(t)*.

Note that in stable state for input $u(t)=U\sin\omega t$ the current has the form $u(t)=U\sin\omega t$ and it is shifted with respect to input by $\frac{\pi}{2}$

Now let us consider the linear electrical circuit shown in Fig. 3.4 with given resistance R, inductance L and source voltage E.

The current $i=i(t)$ and the voltage on the inductance $u=u(t)$ are given by

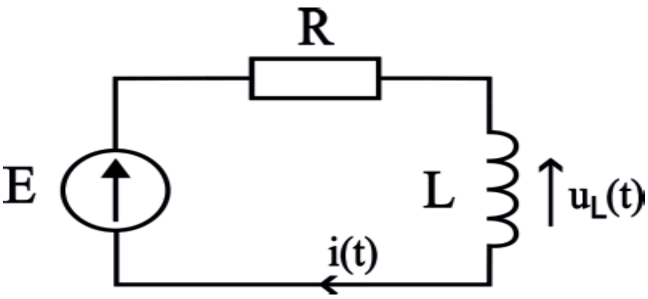

Fig. 3.4. Linear electrical circuit R, L.

(3.3) $$i(t)=\frac{E}{R}(1-e^{-\frac{R}{L}t}),\;\; u_L(t)=e^{-\frac{R}{L}t}$$

and shown in Fig. 3.4

Using the definition (2.5) and (3.3) we obtain

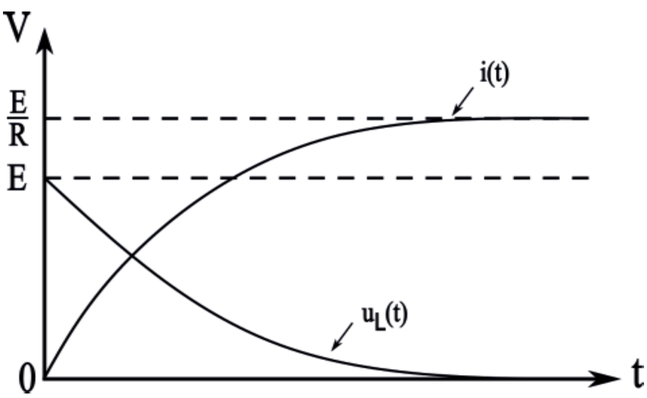

Fig. 3.5. Current *i(t)* and voltage *u(t)*.

$$\cos\varphi(t)=\frac{\int_0^t \frac{E}{R}(1-e^{-\frac{R}{L}t})Ee^{-\frac{R}{L}t}d\tau}{\sqrt{\int_0^t\left(\frac{E}{R}\right)^2(1-e^{-\frac{R}{L}t})^2d\tau}\sqrt{\int_0^t E^2\left(e^{-\frac{R}{L}t}\right)^2d\tau}}=$$

(3.4) $$=\frac{\int_0^\infty (1-e^{-\frac{R}{L}t})e^{-\frac{R}{L}t}dt}{\sqrt{\int_0^\infty (1-e^{-\frac{R}{L}t})^2dt}\sqrt{\int_0^\infty\left(e^{-\frac{R}{L}t}\right)^2dt}}=0$$

since $\int_0^\infty (1-e^{-\frac{R}{L}t})^2dt=\infty$ and $\int_0^\infty e^{-\frac{R}{L}2t}dt$, $\int_0^\infty\left(e^{-\frac{R}{L}t}-e^{-\frac{R}{L}2t}\right)dt$ are finite.

Therefore, the angle between the current $i(t)$ and the voltage $u(t)$ is equal to $\frac{\pi}{2}$

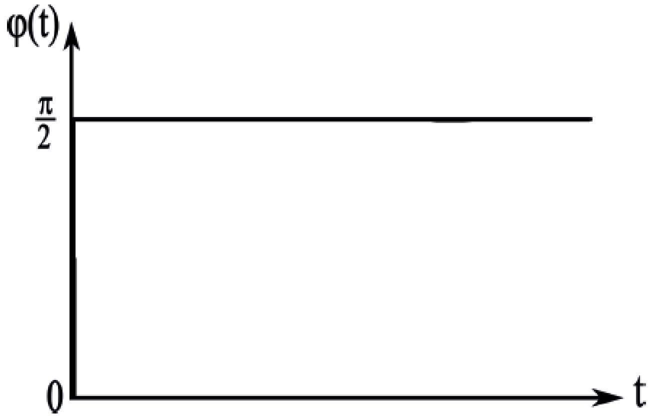

Fig. 3.6. The angle between the voltage *u(t)* and current *i(t)*.

Note that in stable state for input $i(t)=I\sin\omega t$ the voltage has the form $u(t)=\omega LI\cos\omega t$ and it is sifted with respect to input by

From the above considerations follows the following Theorem.

Theorem 1. The angle between the voltage $u(t)$ on the capacitor and its current $i(t)$ is equal to $\frac{\pi}{2}$. The angle between the current $i(t)$ in the coil and its voltage $u(t)$ is equal to $\frac{\pi}{2}$.

The considerations can be extended to more complicated linear electrical circuits.

Dependence of the angles on the locations of inductances and capacities in the structure of the electrical circuits

Consider the linear electrical circuit shown in Fig. 4.1 with given resistances R_1, R_2, inductance *L*, capacitance *C* and source voltage *E*.

Using the Kirchhoff's laws we may write the equations

(4.1) $$E=R_1i+L\frac{di}{dt}+u,\;\; i=\frac{u}{R_2}+C\frac{du}{dt}$$

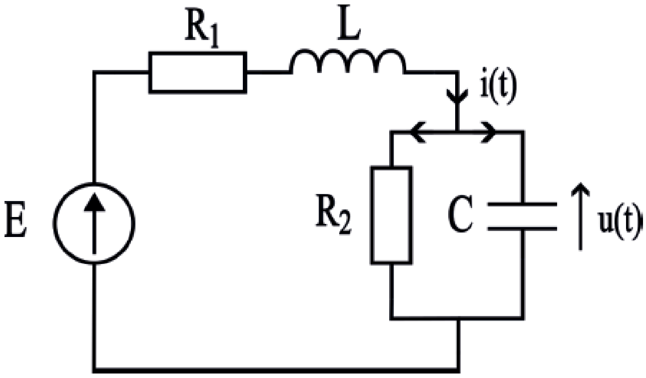

Fig. 4.1. Linear electrical circuit.

which can be written in the form

(4.2a) $$\frac{du}{dt}\begin{bmatrix}u\\ i\end{bmatrix} = A\begin{bmatrix}u\\ i\end{bmatrix} + BE$$

where

(4.2b) $$A = \begin{bmatrix}-\frac{1}{R_2C} & \frac{1}{C}\\ -\frac{1}{L} & -\frac{R_1}{L}\end{bmatrix},\quad B = \begin{bmatrix}0\\ \frac{1}{L}\end{bmatrix}.$$

The electrical circuit is asymptotically stable since the characteristic polynomial

(4.3) $$\det[I_2s - A] = s^2 + \left(\frac{R_1}{L} + \frac{1}{R_2C}\right)s + \frac{R_1}{R_2LC} + \frac{1}{LC}$$

of the matrix A has positive coefficients and its zeros s_1, s_2 are real if the condition

(4.4) $$R_1^2R_2C^2 + L^2 - 2R_1CL - 4R_2LC > 0$$

is satisfied.

The solution to the equation (4.2a) for zero initial conditions has the form

$$\begin{bmatrix}u\\ i\end{bmatrix} = \int_0^t e^{A\tau}BEd\tau = (e^{At} - I)A^{-1}BE =$$

(4.5) $$= [Z_1(e^{s_1t} - 1)s_1^{-1} + Z_2(e^{s_2t} - 1)s_2^{-1}]BE$$

where $Z_1 = \frac{1}{s_1 - s_2}[A - I_2s_2]$, $Z_2 = \frac{1}{s_2 - s_1}[A - I_2s_1]$, s_1, s_2 are the zeros of the polynomial (4.3).

Now let us consider the linear electrical circuit shown in Fig. 4.2 with given resistances R_1, R_2, inductance L, capacitance C and source voltage E. Note that the electrical circuit shown

in Fig. 4.2 differs from the one shown in Fig. 4.1 by the exchange of the inductance L and capacitance C.

Using the Kirchhoff's laws we may write the equations

(4.6) $$E = R_1\left(i + \frac{L}{R_2}\frac{di}{dt}\right) + u + L\frac{di}{dt},\quad C\frac{du}{dt} = i + \frac{L}{R_2}\frac{di}{dt}$$

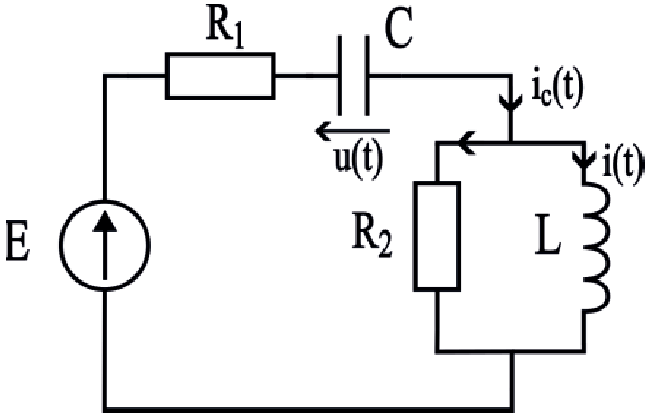

Fig. 4.2. Linear electrical circuit.

which can be written in the form

(4.7a) $$\frac{du}{dt}\begin{bmatrix}u\\ i\end{bmatrix} = A\begin{bmatrix}u\\ i\end{bmatrix} + BE$$

where

(4.7b) $$A = \begin{bmatrix}-\frac{1}{(R_1+R_2)C} & \frac{R_2}{(R_1+R_2)C}\\ -\frac{R_2}{(R_1+R_2)L} & -\frac{R_1R_2}{(R_1+R_2)L}\end{bmatrix},\quad B = \begin{bmatrix}\frac{1}{(R_1+R_2)C}\\ \frac{R_2}{(R_1+R_2)L}\end{bmatrix}.$$

The electrical circuit is asymptotically stable since the characteristic polynomial

(4.8) $$\det[I_2s - A] = s^2 + \frac{L + R_1R_2C}{(R_1+R_2)LC}s + \frac{R_2(R_1+R_2)}{(R_1+R_2)LC}$$

of the matrix A has positive coefficients and its zeros s_1, s_2 are real if the condition

(4.9) $$(L + R_1R_2C)^2 - 4(R_1 + R_2)R_2LC > 0$$

is satisfied

Note that the characteristic polynomials (4.3) and (4.8) are different and the conditions and (4.9) for real roots s_1, s_2 are different.

The solution to the equation (4.7) for zero initial conditions has the form

(4.10) $$\begin{bmatrix}u\\ i\end{bmatrix} = \int_0^t e^{A\tau}BEd\tau = [Z_1(e^{s_1t} - 1)s_1^{-1} + Z_2(e^{s_2t} - 1)s_2^{-1}]BE$$

where $Z_1 = \frac{1}{s_1 - s_2}[A - I_2s_2]$, $Z_2 = \frac{1}{s_2 - s_1}[A - I_2s_1]$, s_1, s_2,are the zeros of the polynomial (4.3).

Concluding remarks

The classical definition of the angles between steady values of the voltages and currents in the electrical circuits with constant values of the resistances, inductances and capacitances has been extended to the transient values of the voltages and currents. The theses have been shown on simple electrical circuits with constant resistances, inductances and capacitances.

It has been shown that the angles depend on the location of the resistances, inductances and capacitances of the electrical circuits. The considerations can be easily extended to the fractional orders linear electrical circuits [5–7, 9].

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